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INDEXIALS AND NORMALIZERS

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Abstract

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MATHEMATICS

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INDEXIALS AND NORMALIZERS

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§ 1. In a number of papers ⁽¹⁻⁷⁾ we have investigated, for finite groups, questions of the existence and conjugacy of subgroups whose orders and indices are relatively prime, as well as conditions for embedding in them subgroups of smaller orders. Properties of finite groups of this kind we called Sylow properties ⁽³⁾, or Π -Sylow properties ⁽⁸⁾, since the classical Sylow theorem corresponds to their particular case, when the set Π of prime divisors of the order of the subgroup under investigation reduces to a single prime number p .

One of the interesting examples of Π -Sylow properties is provided by the p -solvable groups introduced by us in ⁽¹⁾, which subsequently found application in many works by various authors (we shall confine ourselves here to mentioning only two of them ⁽⁹⁾). Another example of this type may be the universal factorization of finite groups found in ^(5,7), which in the particular case of solvable groups turns into the classical factorization of P. Hall.

If the literature on subgroups whose orders and indices are relatively prime has by now become sufficiently extensive, then far fewer investigations have been devoted to questions of the existence of subgroups of arbitrary order. Some criteria of this character are contained in ⁽¹⁰⁾. The factorization theorems of the papers ⁽¹¹⁾ also belong to them.

The method of indexials introduced by us in ⁽¹²⁾ united a large number of previously known criteria for the existence of subgroups, making it possible also to detect subgroups whose order and index are not necessarily relatively prime. The essence of the method of indexials, presented in the most general outline, consists in taking certain divisors of the indices of a certain series of invariant subgroups of a group and then forming their product h , called an indexial.

In the paper ⁽¹³⁾ it was shown that every indexial h of a finite group can be "expanded" in such a way as to obtain a new, "regular" indexial of the form hc , $c \geq 1$, to which there already corresponds a "suitable" subgroup of order hc , where the natural number c is such that the set of its prime divisors is contained in the set of prime divisors of the number h . Thus this theorem also serves as a means for detecting subgroups.

The rudiments of the method of indexials may be discerned in Schur's well-known factorization theorem ⁽¹⁴⁾, in Theorem 9 of the paper ⁽⁴⁾, and in Theorem E1* of the paper ⁽¹⁵⁾. In the present paper, by the method of indexials, we obtain several more criteria for the existence of subgroups in finite groups, adjoining the results obtained in ⁽¹³⁾. In doing so, certain subgroups of the normalizers of the factors of an indexial in the factor groups of the original invariant series of the group are used.

§ 2. In what follows we shall use a number of definitions, notations, and assertions of the paper ⁽¹³⁾: $\mathfrak{G}$ is an arbitrary finite group, whose order we denote by $(\mathfrak{G})$; $\mathfrak{E}$ is the identity subgroup of $\mathfrak{G}$; $\Pi(n)$ is the set of prime divisors of the natural number n ; a Π -number is a natural number a for which $\Pi(a) \subseteq \Pi$, where Π is some nonempty

or the empty set of prime numbers; a Π -divisor of a natural number g is a divisor of g that is a Π -number.

If N is some subsequence of a certain sequence of natural numbers M (in which repetitions are possible), then by $\bar{N}$ we shall denote, in the case where N is nonempty, the product of all terms of N , and in the case where N is empty we put $\bar{N} = 1$.

We shall denote by P_{f_i} the set of terms of the sequence $f_\beta, f_{\beta+1}, \dots, f_\nu$ preceding its term f_i (excluding f_i).

Let now

$$\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \dots \supseteq \mathfrak{G}_\nu = \mathfrak{E}, \quad \nu \geq 1, \quad (R)$$

be a certain series of normal divisors of the group $\mathfrak{G}$, with or without repetitions, and with a sequence of indices $h_1, h_2, \dots, h_\nu$.

Definition of an indexial. Suppose that on a sequence of natural numbers $W = \{\beta, \beta + 1, \dots, \nu\}$, $1 \leq \beta \leq \nu$, there is defined a certain single-valued function f such that: a) to each $i \in W$ it assigns a certain subgroup $\mathfrak{F}_i/\mathfrak{G}_i$, of order $(\mathfrak{F}_i/\mathfrak{G}_i) = f_i$, of the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$, and b) all subgroups of $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ conjugate to $\mathfrak{F}_i/\mathfrak{G}_i$ in the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ are already conjugate to $\mathfrak{F}_i/\mathfrak{G}_i$ by means of an element from $\mathfrak{G}_{i-1}/\mathfrak{G}_i$.

Then the product

$$(\mathfrak{F}_\beta/\mathfrak{G}_\beta)(\mathfrak{F}_{\beta+1}/\mathfrak{G}_{\beta+1}) \dots (\mathfrak{F}_\nu/\mathfrak{G}_\nu) = f_\beta f_{\beta+1} \dots f_\nu = h$$

we shall call an **indexial of the series (R) of the group $\mathfrak{G}$** , or, more briefly, an **indexial of the group $\mathfrak{G}$** .

Symbolically, taking into account the way it is obtained, we shall write the indexial as $(h)_{R,f}$, or in those cases where the form of the function f is evident, as $(h)_R$. The groups $\mathfrak{F}_\beta/\mathfrak{G}_\beta, \mathfrak{F}_{\beta+1}/\mathfrak{G}_{\beta+1}, \dots, \mathfrak{F}_\nu/\mathfrak{G}_\nu$ we shall call the **factors of the indexial $(h)_R$** . We shall call the indexial $(h)_R$ **regular** if the group $\mathfrak{G}$

has at least one subgroup $\mathfrak{H}$ of order h , contained in $\mathfrak{G}_{\beta-1}$, for which, for every $i \in W$, the condition

$$\mathfrak{F}_i = [\mathfrak{H} \cap \mathfrak{G}_{i-1}] \mathfrak{G}_i$$

is satisfied.

Corollary of Theorem 6 from (13). For every indexial $(h)_R$ of the group $\mathfrak{G}$ there exists a regular indexial

$$(ch)_R = c_\beta f_\beta c_{\beta+1} f_{\beta+1} \cdots c_\nu f_\nu,$$

$$\Pi(c_i) \subseteq \Pi(\overline{P}_{f_i}), \quad i \in W,$$

in which each of its factors $\mathfrak{C}_i/\mathfrak{G}_i$, $i \in W$, is an extension of the subgroup $\mathfrak{F}_i/\mathfrak{G}_i$ by means of a special (nilpotent) group of order c_i . Consequently, in this case $\mathfrak{G}$ has at least one subgroup of order

$$c_\beta f_\beta c_{\beta+1} f_{\beta+1} \cdots c_\nu f_\nu$$

(note that from $\Pi(c_\beta) \subseteq \Pi(\overline{P}_{f_\beta})$ it follows that $c_\beta = 1$).

§ 3. We introduce still the following new notation and definitions. Let a subgroup $\mathfrak{N}_i/\mathfrak{G}_i$ of order $n_i f_i$, $i \in W$, for $i = \beta$ coincide with $\mathfrak{F}_\beta/\mathfrak{G}_\beta$, and for $i > \beta$ with the normalizer of the subgroup $\mathfrak{F}_i/\mathfrak{G}_i$ in the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ (whence it follows that $n_\beta = 1$).

Definition 1. Suppose that for each $i \in W$ a subgroup $\mathfrak{M}_i/\mathfrak{G}_i$ of the group $\mathfrak{N}_i/\mathfrak{G}_i$ is given, coinciding for $i = \beta$ with $\mathfrak{F}_\beta/\mathfrak{G}_\beta$ and such that all subgroups of the group $\mathfrak{N}_i/\mathfrak{G}_i$ conjugate to $\mathfrak{M}_i/\mathfrak{G}_i$ in the group $\mathfrak{G}_{\beta-1}/\mathfrak{G}_i$ are already conjugate to $\mathfrak{M}_i/\mathfrak{G}_i$ in $\mathfrak{N}_i/\mathfrak{G}_i$. Let $\mathfrak{M}_i^*/\mathfrak{G}_i$, $i \in W$, be the normalizer of the subgroup $\mathfrak{M}_i/\mathfrak{G}_i$ in the group $\mathfrak{N}_i/\mathfrak{G}_i$. Define the system of numbers m_i^* , $i \in W$, by means of the equalities

$$(\mathfrak{M}_i^* \mathfrak{F}_i/\mathfrak{G}_i) = m_i^* f_i.$$

Then the system of subgroups $\mathfrak{M}_i/\mathfrak{G}_i$ and the system of numbers m_i^* , $i \in W$, will be called **corresponding to each other systems of supplementary subgroups and supplementary multipliers of the indexial $(h)_R$** . Obviously, for a given indexial there may exist several supplementary systems of subgroups and, consequently, also of multipliers.

From Definition 1 it follows that $m_\beta^* = 1$.

As the subgroup $\mathfrak{M}_i/\mathfrak{G}_i$, $i > \beta$, one may take, for example, $\mathfrak{N}_i/\mathfrak{G}_i$ itself or any of its Sylow subgroups.

We shall present the results obtained by us.

Theorem 1. If a group $\mathfrak{G}$ has $(h)_R$ with some system of supplementary factors m_i^* , $i \in W$, then $\mathfrak{G}$ contains at least one subgroup of order

$$m_\beta^* f_\beta m_{\beta+1}^* f_{\beta+1} \cdots m_\nu^* f_\nu.$$

Theorem 2. If a group $\mathfrak{G}$ has an indexial $(h)_R$, then it contains at least one subgroup of order

$$n_\beta f_\beta n_{\beta+1} f_{\beta+1} \dots n_\nu f_\nu.$$

Theorem 3. If a group $\mathfrak{G}$ has an indexial $(h)_R$ and every subgroup $\mathfrak{F}_i/\mathfrak{G}_i$ for $i > \beta$ coincides with its normalizer in the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$, then the indexial $(h)_R$ is proper and, consequently, $\mathfrak{G}$ contains at least one subgroup of order h .

Definition 2. Let the group $\mathfrak{A}$ have subgroups $\mathfrak{B}$ and $\mathfrak{C} = \mathfrak{C}^{(0)}$, with $\mathfrak{B} \supseteq \mathfrak{C}$. Then a series of subgroups

$$\mathfrak{C}^{(1)} \subseteq \mathfrak{C}^{(2)} \subseteq \dots,$$

where $\mathfrak{C}^{(i)}$ is the normalizer of $\mathfrak{C}^{(i-1)}$ in the subgroup $\mathfrak{B}$, $i = 1, 2, \dots$, will be called the **normalizer series** of $\mathfrak{C}$ in the subgroup $\mathfrak{B}$.

Theorem 4. Let the group $\mathfrak{G}$ have an indexial $(h)_R$, and let, for each $i \in W$, a subgroup $\mathfrak{M}_i/\mathfrak{G}_i$ of the group $\mathfrak{G}_{i-1}/\mathfrak{G}_i$ be given, which for $i = \beta$ coincides with $\mathfrak{F}_\beta/\mathfrak{G}_\beta$, and for $i > \beta$ satisfies the following conditions:

1. $\mathfrak{M}_i/\mathfrak{G}_i$ belongs to some term $\mathfrak{F}_i^{(j_i)}/\mathfrak{G}_i$ of the normalizer series of the subgroup $\mathfrak{F}_i/\mathfrak{G}_i$ in the subgroup $\mathfrak{G}_{i-1}/\mathfrak{G}_i$.
2. All subgroups of $\mathfrak{F}_i^{(j_i)}/\mathfrak{G}_i$ conjugate with $\mathfrak{M}_i/\mathfrak{G}_i$ in $\mathfrak{G}_{\beta-1}/\mathfrak{G}_i$ are conjugate with $\mathfrak{M}_i/\mathfrak{G}_i$ already in $\mathfrak{F}_i^{(j_i)}/\mathfrak{G}_i$.

Then $\mathfrak{G}$ has at least one subgroup of every order of the form

$$(\mathfrak{F}_\beta/\mathfrak{G}_\beta)(\mathfrak{M}_{\beta+1}^{(k_{\beta+1})}\mathfrak{F}_{\beta+1}^{(j_{\beta+1}-1)}/\mathfrak{G}_{\beta+1}) \dots (\mathfrak{M}_\nu^{(k_\nu)}\mathfrak{F}_\nu^{(j_\nu-1)}/\mathfrak{G}_\nu),$$

where $\mathfrak{M}_i^{(k_i)}/\mathfrak{G}_i$, $i > \beta$, $i \in W$, is some term of the normalizer series of the subgroup $\mathfrak{M}_i/\mathfrak{G}_i$ in the subgroup $\mathfrak{F}_i^{(j_i)}/\mathfrak{G}_i$.

Theorem 5. If a group $\mathfrak{G}$ has an indexial $(h)_R$, then it has at least one subgroup of every order of the form

$$(\mathfrak{F}_\beta/\mathfrak{G}_\beta)(\mathfrak{F}_{\beta+1}^{(j_{\beta+1})}/\mathfrak{G}_{\beta+1}) \dots (\mathfrak{F}_\nu^{(j_\nu)}/\mathfrak{G}_\nu),$$

where all $j_i \geq 1$, $i \in W$, $i > \beta$.

Theorem 6. If a group $\mathfrak{G}$ has an indexial $(h)_R$, and if h_i/n_i for each $i \in W$ is divisible by the greatest $\Pi(\overline{P}_{f_i})$ -divisor of the number h_i , then the indexial $(h)_R$ is proper and, consequently, $\mathfrak{G}$ contains at least one subgroup of order h .

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REFERENCES

1. S. A. Chunihin, DAN, 55, No. 6, 481 (1947).
2. S. A. Chunihin, DAN, 58, No. 7, 1295 (1947); 59, No. 3, 443 (1948); 60, No. 5, 773 (1948); 66, No. 2, 165 (1949); 69, No. 6, 735 (1949); 77, No. 6, 973 (1951); 83, No. 5, 663 (1952); 86, No. 1, 27 (1952); 91, No. 3, 461 (1953); Matem. sborn., 25 (67), No. 3, 321 (1949).
3. S. A. Chunihin, DAN, 73, No. 1, 29 (1950).
4. S. A. Chunihin, Matem. sborn., 33 (75), No. 1, 141 (1953).
5. S. A. Chunihin, DAN, 97, No. 6, 977 (1954).
6. S. A. Chunihin, DAN, 103, No. 3, 377 (1955).
7. S. A. Chunihin, Matem. sborn., 39 (81), No. 4, 465 (1956).
8. S. A. Chunihin, *Subgroups of finite groups*, Minsk, 1964.
9. P. Hall, G. Higman, Proc. London Math. Soc., (3) 6, No. 21, 1 (1956); W. Feit, J. Thompson, Pacific J. Math., 13, No. 3, 773 (1963).
10. S. A. Chunihin, Matem. sborn., 40, No. 1, 39 (1933); 4 (46), No. 3, 521 (1938); Matem. sborn., 62 (104), No. 1, 76 (1963).
11. S. A. Chunihin, DAN, 108, No. 3, 397 (1956); Matem. sborn., 54 (96), No. 2, 237 (1961).
12. S. A. Chunihin, DAN, 124, No. 2, 243 (1958).
13. S. A. Chunihin, Matem. sborn., 55 (97), No. 2, 101 (1961).
14. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, 1, 1937.
15. P. Hall, Proc. London Math. Soc., (3) 6, No. 22, 286 (1956).

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