

ON THE BEHAVIOR OF METRIZABILITY UNDER QUOTIENT $\backslash(s\backslash)$ -MAPPINGS

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1966

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Abstract

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UDC 519.50+519.54

MATHEMATICS

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ON THE BEHAVIOR OF METRIZABILITY UNDER QUOTIENT s -MAPPINGS*

(Presented by Academician P. S. Aleksandrov on May 20, 1965)

The present paper is devoted to the study of quotient spaces of metrizable spaces. Its main results are Theorems 1 and 2.

Theorem 1. *Let $f : X \rightarrow Y$ be a quotient s -mapping of a metric space X onto a regular space Y with the first axiom of countability. If the space Y is separable,** then it is also metrizable.*

Proof. Let $\Phi = \{y_1, \dots, y_n, \dots\}$ be some countable everywhere dense subset of Y . Put $X_1 = [f^{-1}\Phi]$ and $f_1 = f|X_1$. It is possible to verify that $fX_1 = Y$. By virtue of the separability of the mapping f , the subspace X_1 has a countable base. Let $\omega = \{U_i \mid i = 1, \dots\}$ be some such base. Suppose that the union of any finite number of elements of ω is also contained in ω . We shall prove that the system

$$\gamma = \{\text{Int}[f_1 U_i] \mid U_i \in \omega\}$$

forms a base in Y . Take an arbitrary point $y_0 \in Y$ and an arbitrary neighborhood $Oy_0 \ni y_0$. Consider a neighborhood $O_1 y_0$ such that $y_0 \in O_1 y_0 \subset [O_1 y_0] \subset Oy_0$. In ω there is a sequence $\{U_{i_k} \mid k = 1, \dots\}$ of sets open in X_1 such that: 1) $U_{i_k} \subset f_1^{-1} O_1 y_0$,

$$k = 1, 2, \dots; \quad 2) \quad U_{i_{k+1}} \supset U_{i_k}, \quad k = 1, 2, \dots; \quad 3) \quad f_1^{-1} y_0 \subset \bigcup_{k=1}^{\infty} U_{i_k}.$$

At the point y_0 take a base $\{W_k \mid k = 1, 2, \dots\}$ such that $W_k \subset O_1 y_0$ and $W_{k+1} \subset W_k$, $k = 1, \dots$. We shall prove that there is a number k_0 such that $[f_1 U_{i_{k_0}}] \supset W_{k_0}$. Suppose the contrary; then $(W_k \setminus [f_1 U_{i_k}]) \cap \Phi \neq \emptyset$, where $k = 1, 2, \dots$. Construct a sequence $P = \{y_{n_k} \mid y_{n_k} \in (W_k \setminus [f_1 U_{i_k}]) \cap \Phi\}$ which converges to the point y_0 . From the fact that $y_{n_k} \in W_k \setminus f_1 U_{i_k}$ and $U_{i_1} \subset \dots \subset U_{i_k} \subset \dots$, it follows that each set U_{i_k} intersects only finitely many sets

$$f_1^{-1}y_{n_{j_1}}, \dots, f_1^{-1}y_{n_{j_{s(k)}}}$$

from $f_1^{-1}P$. Hence the set

$$\widetilde{U}_{i_k} = U_{i_k} \setminus f_1^{-1}P = U_{i_k} \setminus \bigcup_{j=1}^{s(k)} f_1^{-1}y_{n_{j_i}}$$

is open in X_1 . From the fact that $f_1^{-1}P \cap f_1^{-1}y_0 = \emptyset$ and

$$\bigcup_{k=1}^{\infty} U_{i_k} \supset f_1^{-1}y_0,$$

it follows that

$$\bigcup_{k=1}^{\infty} \widetilde{U}_{i_k} \supset f_1^{-1}y_0 \quad \text{and} \quad \bigcup_{k=1}^{\infty} \widetilde{U}_{i_k} \cap f_1^{-1}P = \emptyset.$$

We have obtained that not a single point of $f_1^{-1}y_0$ is contained in $[f_1^{-1}P]_{X_1}$. But, since $X_1 = [f^{-1}\Phi]_X$, $P \subset \Phi$, and by virtue of the quotient character of the mapping f , we obtain that

$$[f_1^{-1}P]_{X_1} \cap f_1^{-1}y_0 \neq \emptyset,$$

a contradiction. Hence there is a number k_0 such that $[f_1 U_{i_{k_0}}] \supset W_{k_0}$, and this condition is equivalent to

$$\text{Int}[f_1 U_{i_{k_0}}] \ni y_0.$$

From the fact that $U_{i_{k_0}} \subset f_1^{-1}O_1 y_0$ and

* A mapping $f : X \rightarrow Y$ is called an s -mapping (or separable) if for every point $y \in Y$ the set $f^{-1}y$ is a subspace with a countable base.

** A space Y is separable if it contains some everywhere dense countable set.

$[O_1 y_0] \subset O y_0$, it follows that $\text{Int}[f_1 U_{i_{k_0}}] \subset O y_0$. This completes the proof of Theorem 1.

Corollary (Stone). Let $f : X \rightarrow Y$ be a quotient mapping of a separable metric space X onto a regular space Y with the first axiom of countability. Then Y is a separable metrizable space.

In Theorem 1 one cannot discard the assumption that the space Y is separable, since from results of V. Ponomarev it follows that every topological space with a point-countable base (among them there are also nonmetrizable ones) is an open s -image of some strongly paracompact metric space (see ⁽⁶⁾). Since there exist nonmetrizable spaces that are closed images of metric spaces with a countable base, the requirement of the first axiom of countability is necessary. Thus Theorem 1 may be regarded as a final result. For the same reasons Theorem 2 should also be recognized as final. Theorem 2 is a strengthening of Stone's well-known theorem on open s -mappings and of one result of A. Arhangel'skii.

A. Arhangel'skii has recently introduced the notion of a space of point-countable type ⁽³⁾.

Definition. A space X is called a **space of point-countable type** if an arbitrary point $x \in X$ is contained in some bicomact $F \subseteq X$ whose character is countable.*

Theorem 2. Let $f : X \rightarrow Y$ be a quotient s -mapping of a locally separable metric space X onto a regular weakly paracompact space Y . If Y is a space of point-countable type, then it is metrizable and satisfies the local second axiom of countability.**

On the way to the proof of Theorem 2 we shall obtain a number of auxiliary assertions which may perhaps be of independent interest. In parallel, a theorem on quotient uniform s -mappings is proved.

Lemma. (A. Arhangel'skii). Let $f : X \rightarrow Y$ be a quotient mapping of a topological space X onto a bicomactum Y . For every open cover $\eta = \{U_\alpha\}$ of the space X there exists in Y such a finite set of points $\Phi = \{y_1, \dots, y_n\}$ that, whatever the point $y \in Y$, for some point $y' \in \Phi$ and some element $U_{\alpha'} \in \eta$, $f_y^{-1} \cap U_{\alpha'} \neq \emptyset$ and $f^{-1}y' \cap U_{\alpha'} \neq \emptyset$.

Proof. Suppose the contrary. Let $\Phi_1 = \{y_1\}$, where y_1 is some point of Y . There is a point $y_2 \in Y$ such that the condition $\eta f^{-1}\Phi_1 \cap f^{-1}y_2 = \emptyset$ is satisfied. Put $\Phi_2 = \{y_1, y_2\}$. Suppose the set $\Phi_k = \{y_1, \dots, y_k\}$ has been constructed. By the assumption, there is a point $y_{k+1} \in Y$ such that $\eta f^{-1}\Phi_k \cap f^{-1}y_{k+1} = \emptyset$. Put $\Phi_{k+1} = \Phi_k \cup y_{k+1} = \{y_1, \dots, y_k, y_{k+1}\}$. Let us have constructed the sets $\Phi_1, \dots, \Phi_k, \dots$, $\Phi_k \subset \Phi_{k+1}$, where $k = 1, 2, \dots$. Let $\Phi = \bigcup_{k=1}^{\infty} \Phi_k$. For any two points $y', y'' \in \Phi$ we have

$$\eta f^{-1}y' \cap f^{-1}y'' = \emptyset, \quad \eta f^{-1}y'' \cap f^{-1}y' = \emptyset. \quad (\text{A})$$

* A bicomactum $F \subset X$ is called a bicomactum of countable character (in X) if there exists a countable system of open sets $\varphi = \{U_n\}$ such that for every $U \supset F$ there is $U_n \in \varphi$ for which $F \subset U_n \subseteq U$. The spaces of point-countable type include all metric spaces, all locally bicomact spaces, and even all spaces complete in the sense of Čech.

** Earlier A. Arhangel'skii proved this proposition for the case of a locally compact space Y under the additional assumption that the mapping f is pseudo-open.

Y is bicomact. There is a point $y_0 \in Y$ such that $y_0 \in [\Phi]$. Put $\Phi_0 = \Phi \setminus y_0$. Φ_0 is not closed in Y ; therefore, by virtue of the quotient nature of the mapping, the set $f^{-1}\Phi_0$ is not closed in X , i.e., there is a point $x_0 \in [f^{-1}\Phi_0] \setminus f^{-1}\Phi_0$. Take some $U_{\alpha'} \in \eta$ such that $x_0 \in U_{\alpha'}$. From condition (A) it follows that $U_{\alpha'}$ meets no more than one $f^{-1}y_{k_0}$. Hence the open set

$$\Gamma = U_{\alpha'} \cap (X \setminus f^{-1}y_{k_0})$$

does not meet $f^{-1}\Phi_0$, and this means that $x_0 \notin [f^{-1}\Phi_0]$, a contradiction. Lemma 2 is completely proved.

Theorem 3. *Let $f : X \rightarrow Y$ be a quotient uniform s -mapping of a metrizable space X onto a bicomact Y . Then the space Y is metrizable.*

Proof. Take the system

$$\left\{ \gamma_n = \{U_\alpha^n \mid \text{diam } U_\alpha^n < \frac{1}{n}\} \right\}$$

of locally finite coverings which forms a base in X . Take some γ_n . There is a set

$$\Phi_n = \{y_1^n, \dots, y_{k(n)}^n\}$$

satisfying the conditions of A. Arhangel'skii's lemma. Take the system

$$\bar{\gamma}_n = \{U_\alpha^n \mid U_\alpha^n \cap f^{-1}\Phi_n \neq \emptyset\}.$$

Since f is an s -mapping, the system $\bar{\gamma}_n$ is countable. We shall show that the system

$$\Omega = \bigcup_{n=1}^{\infty} f\bar{\gamma}_n$$

is a network in the space Y .

Take arbitrary y_0 and Oy_0 from Y . Since f is uniform, we have

$$\rho(f^{-1}y_0, X \setminus f^{-1}Oy_0) = r > 0$$

(see (6)). Choose an index n_0 such that $1/n_0 < r/2$. By the condition, there exist $y' \in \Phi_{n_0}$ and $U_{\alpha'}^{n_0} \in \bar{\gamma}_{n_0}$ such that

$$f^{-1}y_0 \cap U_{\alpha'}^{n_0} \neq \emptyset.$$

Since

$$\text{diam } U_{\alpha'}^{n_0} < 1/n_0 < r/2,$$

we have

$$U_{\alpha'}^{n_0} \subset f^{-1}Oy_0.$$

Then

$$y_0 \in fU_{\alpha'}^{n_0} \subset Oy_0,$$

and this means that Ω is a countable network in the space Y , and on the basis of Theorem 1 from (2), Theorem 3 is proved.

Lemma 1. *Let $f : X \rightarrow Y$ be a quotient s -mapping of a locally separable metrizable space X onto a bicomact Y . Then the space Y is metrizable.*

Proof. Since X locally satisfies the second axiom of countability, it decomposes into some number of discrete sets

$$\omega = \{M_\lambda \mid \lambda \in \theta\},$$

where each M_λ has a countable base. The system $\omega = \{M_\lambda\}$ forms a covering of the space X . On the basis of A. Arhangel'skii's lemma, there is a finite set of points

$$\Phi = \{y_1, \dots, y_n\}$$

such that

$$f(\omega f^{-1}\Phi) = Y.$$

Denote

$$X_1 = \omega f^{-1}\Phi$$

and $f_1 = f \mid X_1$. Since f is an s -mapping, X_1 has a countable base. By the continuity of the mapping f_1 , the space Y has a countable network and, consequently, is metrizable (see Theorem 1 from (2)).

Lemma 2*. *If a bicomact Φ has countable character in a T_2 -space X , and a point $x_0 \in \Phi \subset X$ has countable character in the subspace Φ , then the point x_0 has countable character also in the space X .*

Proof. Let

$$\varphi = \{U_n \mid n = 1, \dots\}$$

be a countable system of open sets in X such that for any open set $U \supset \Phi$ there is U_n for which

$$\Phi \subset U_n \subset U.$$

Let, further,

$$\pi = \{\Gamma_n\}$$

be a countable system of open sets in Φ , forming a base at the point x_0 . For each point $x \neq x_0$ of Φ there exists a neighborhood Ox (open in X) such that

$$x_0 \notin [Ox].$$

Obviously,

$$\gamma_n = \{\Gamma_n, Ox \mid x \in \Phi, x \neq x_0\}$$

is a covering of the bicomact Φ . Choose a finite subcovering

$$\omega_n = \{\Gamma_n, Ox_1^n, \dots, Ox_{m(n)}^n\}.$$

Denote

$$M_n = \bigcup_{i=1}^{m(n)} O x_i^n.$$

We have

$$x_0 \notin [M_n].$$

Put

$$V_n = U_n \setminus \bigcup_{i=1}^n [M_i] = U_n \cap \left(X \setminus \bigcup_{i=1}^n [M_i] \right).$$

We shall show that the system

$$\Omega = \{V_n \mid n = 1, \dots\}$$

forms a base at the point x_0 in the space—

* For the case of completely regular spaces this lemma was first established by A. Arhangel'skii by a method essentially using complete regularity—the passage to bicomact Hausdorff extensions.

space X . Let U be any open set in X containing the point x_0 . Choose Γ_{n_0} and M_{n_0} such that $\Gamma_{n_0} \subset U$ and $\Phi \subset \Gamma_{n_0} \cup M_{n_0}$. Then $\Phi \subset U \cup M_{n_0}$. There exists an open set U_{n_1} ($n_1 \geq n_0$) such that $U_{n_1} \subset M_{n_0} \cup U$. Then

$$V_{n_1} = U_{n_1} \setminus \bigcup_{i=1}^{n_1} [M_i] \subset U_{n_1} \setminus M_{n_0} \subset U.$$

This completes the proof of Lemma 2.

Proof of Theorem 2. Take some point $y_0 \in Y$. In Y there is a bicomactum Φ_0 of countable character such that $y_0 \in \Phi_0$. On the basis of Lemma 1, the bicomactum Φ_0 is metrized, and on the basis of Lemma 2 the point y_0 has countable character in the space Y . Hence Y satisfies the first axiom of countability. Since X locally satisfies the second axiom of countability, by a known theorem of P. S. Aleksandrov (see ⁽¹⁾), it decomposes into the sum of a certain number of discrete sets $\{M_\lambda\}$, where each M_λ has a countable base. Take any point $y_0 \in Y$. By virtue of the separability of the mapping, the set $f^{-1}y_0$ intersects no more than a countable number of sets $M_{\lambda_1}, \dots, M_{\lambda_n}, \dots$, i.e.

$$f_{y_0}^{-1} \subset \bigcup_{i=1}^{\infty} M_{\lambda_i}.$$

It is known that

$$\text{Int } f \left(\bigcup_{i=1}^{\infty} M_{\lambda_i} \right) = Y_1 \ni y_0$$

and Y_1 is separable. Denote $X_1 = f^{-1}Y_1$ and $f_1 = f|X_1$. Since f_1 is a quotient s -mapping of the metric space X_1 onto the separable space Y_1 with the first axiom of countability, it follows from Theorem 1 that Y_1 has a countable base.

Since Y is a regular weakly paracompact space with the local second axiom of countability, by another theorem of P. S. Aleksandrov (see Theorem 11 in ⁽¹⁾), it is metrizable.

From Theorem 2 it follows that

Corollary. *Let $f : X \rightarrow Y$ be an almost open s -mapping* of a locally separable metric space X onto a regular weakly paracompact space Y . Then Y is metrizable and locally satisfies the second axiom of countability.*

Remark. In Theorem 2 the requirement of weak paracompactness can be weakened to the requirement: into every cover one can inscribe a point-countable cover.

The author expresses gratitude to A. V. Arhangel'skii for posing a number of problems and for providing the opportunity to become acquainted with his unpublished work.

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Received
12 V 1965

REFERENCES

- ¹ P. S. Aleksandrov, P. S. Urysohn, On compact topological spaces, *Tr. po topologii i drugim oblastiam matematiki*, 2, 1951.
- ² A. Arhangel'skii, *DAN*, 126, No. 2 (1959).
- ³ A. Arhangel'skii, *DAN*, 151, No. 4 (1963).
- ⁴ A. Arhangel'skii, *DAN*, 147, No. 5 (1962).
- ⁵ A. N. Stone, *Proc. Am. Math. Soc.*, 7, 690 (1956).
- ⁶ V. I. Ponomarev, *Bull. Polish Acad. Sci.*, 8, No. 3 (1960).

* A mapping f of a space X onto Y is called almost open if for an arbitrary point $y \in Y$ there is a point $x \in f^{-1}y$ having a base of open sets each of whose images is open (see ⁽⁴⁾).

Note: Figure translations are in progress. See original paper for figures.

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