

ON THE THEORY OF LOCALLY FINITE GROUPS

V. P. Shunkov

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.62902>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 512.41

MATHEMATICS

V. P. SHUNKOV

ON THE THEORY OF LOCALLY FINITE GROUPS

(Presented by Academician A. I. Mal' tsev on 15 X 1965)

§ 1. In accordance with the work ⁽¹⁾, a subgroup H of a group G is called **2-infinitely isolated** in G if, from the fact that the centralizer of a nonidentity element of H in G contains at least one involution from G and has infinite intersection with H , it follows that it belongs to the subgroup H .

In the present note we study a locally finite group possessing a maximal complete abelian 2-subgroup whose normalizer is a 2-infinitely isolated subgroup. The theorem proved makes it possible to reduce the solution of the minimality problem ^(2,3) for locally finite groups to the case when its Sylow 2-subgroup is finite.

§ 2. **Definition.** A subgroup H of a group G is **2-mutually simple with all its conjugates** if it has intersection with any subgroup conjugate to it in G , distinct from it, in a subgroup containing no 2-elements. Analogously one may introduce the notion of π -mutual simplicity of a subgroup in a group, where π is a certain set of prime numbers. As in ⁽¹⁾, an involution with finite centralizer in the group will be called **almost regular**.

Theorem. *If in a locally finite group G the normalizer N of some maximal complete abelian 2-subgroup A is a 2-infinitely isolated subgroup, then one of the following assertions holds:*

- 1) *some almost regular involution in H is finitely approximable in the group G ;*
- 2) *the group G is isomorphic to a group of type $PGL(2, K)$ over a field K of odd characteristic;*
- 3) $N = G$;
- 4) *the group G is a Frobenius group.*

If the subgroup N possesses an almost regular involution, then, in view of the main result of ⁽¹⁾, one of assertions 1) and 2) holds. Further, the proof of the theorem is based on the assumption that, in a group satisfying the conditions

of the theorem, none of the possibilities 1)–4) is realized. This part of the proof rests on the following lemmas.

Lemma 1. *All involutions in the group G are conjugate. All involutions from N are conjugate in N and are contained in the subgroup A .*

Lemma 2. *Let G be a certain periodic group, H its proper subgroup, coinciding with its normalizer in G , and 2-mutually simple with all its conjugates.*

If in the subgroup H there exists an elementary abelian 2-subgroup of rank 2, then for some element $h \neq 1$ from H of odd order there is an involution $t \in G$ such that $t \notin H$ and $t^{-1}ht = h^{-1}$.

Proof. From the condition of the lemma follows the existence of an involution $t_1 \in G$ not belonging to the subgroup H . Further, by the condition of the lemma, the subgroup H has two involutions i and j , commuting with each other, and $i \neq j$.

Consider two elements $a = it_1$, $b = jt_1$. They are, evidently, of odd order, since otherwise, by the condition of the lemma, $t_1 \in H$, which is impossible. It follows that the involution t_1 is conjugate to i in the subgroup $\{a\}\lambda\{t_1\}$ and to the involution j in the subgroup $\{b\}\lambda\{t_1\}$. Consequently, for some natural numbers k and r we shall have $a^{-k}ia^k = t_1$, $b^{-r}jb^r = t_1$. Hence $a^{-k}ia^k = b^{-r}jb^r$, $b^ra^{-k}ia^kb^{-r} = j$.

By the condition of the lemma, $h_1 = a^kb^{-r} \in H$. Since $i \neq j$, $h_1 \neq 1$. From $h_1 = a^kb^{-r}$ we obtain $a^k = h_1b^r$. We transform the left and right sides of this equality by means of the involution t_1 : $t_1^{-1}a^kt_1 = t_1^{-1}h_1t_1t_1^{-1}b^rt_1$, or $a^{-k} = t_1^{-1}h_1t_1b^{-r}$. On the other hand, $a^{-k} = b^{-r}h_1^{-1}$. From the last two equalities we obtain $b^{-r}h_1^{-1} = t_1^{-1}h_1t_1b^{-r}$, or $b^rt_1^{-1}h_1t_1b^{-r} = h_1^{-1}$. The element $i_1 = b^rt_1^{-1}$ is an involution, since $i_1 \in \{b^r\}\lambda\{t_1\}$.

Let $i_1 \notin H$. Then the order of the element h_1 is odd. Indeed, suppose that in the subgroup $\{h_1\}$ there exists an involution i_2 . Evidently, $i_1i_2 = i_2i_1$. Since $i_2 \in H$, by the conditions of the lemma, $i_1 \in H$, which is impossible. It follows that the pair of elements h_1 , i_1 satisfies all the requirements of the lemma.

Let $i_1 \in H$. Since $j \in H$ and $j = bt_1^{-1}$, we have $i_1j^{-1} = b^rt_1^{-1}t_1b^{-1} = a^{k-1} \in H$. If $i_2 = i$, $a^{k-1} \neq 1$, and the pair of elements a^{-1} , t_1 satisfies all the requirements of the lemma.

Let $i_1 = j$, i.e., $r = 1$. Then consider the involution $i_2 = a^kt_1^{-1} \in \langle h_1 \rangle \lambda \langle i_1 \rangle$. Since $i_2 \in H$, $i \in H$ and $i = at_1^{-1}$, it follows that $i_2i^{-1} = a^kt_1^{-1}t_1a^{-1} = a^{k-1} \in H$. If $i_2 \neq i$, $a^{k-1} \neq 1$, and the pair of elements a^{k-1} , t_1 satisfies all the requirements of the lemma.

It remains to consider the case when $k = r = 1$. Since $h_1 = a^kb^{-r}$, for $k = r = 1$ we have $h_1 = ab^{-1} = itt^{-1}j^{-1} = ij^{-1}$. By the condition of the lemma, the involutions i and j commute and $i \neq j$, but then the equality $h_1^{-1}ih_1 = j$ is impossible. The contradiction obtained indicates that at least one of the numbers k and r is not congruent to unity modulo the corresponding order of

one of the elements a and b , and, as was shown above, in this case the assertion of the lemma is true. The lemma is proved.

Lemma 3. *The subgroup N is a semidirect product of a Hall subgroup T and some locally cyclic subgroup $D \neq 1$, and every nontrivial element of D induces a regular automorphism in T .*

The proof of the lemma is based on the definition of a 2-infinitely isolated subgroup, Lemmas 1 and 2, and Burnside's theorem ⁽⁴⁾.

Lemma 4. *If the group G satisfies the conditions of Lemma 3, then a Sylow 2-subgroup S of the group G is elementary abelian, and the centralizer of any involution from S in G is contained in the centralizer of the subgroup S in G .*

In the proof of the lemma, Lemmas 1 and 3 are used, together with Thompson's theorem on a finite group possessing a regular automorphism of prime order ⁽⁵⁾, and the Higman-Hall theorem on the length of the upper central series of a finite nilpotent group possessing an automorphism of prime order ⁽⁶⁾.

The concluding part of the proof of the theorem is based on a result of W. Feit on the abstract characterization of groups of type $SL(2, 2^a)$, $a > 1$ ⁽⁷⁾.

We turn to the discussion of the minimality problem in locally finite groups. The problem is formulated as follows ⁽³⁾: is a locally finite group with the minimality condition for subgroups extremal?

Let G be a locally finite group with the minimality condition for subgroups, not extremal. As is not difficult to show, without violating the generality of the argument one may assume that G is an infinite simple group and all proper subgroups of G are extremal. Since a finite group of odd order is solvable ⁽⁸⁾, and a locally solvable group with the minimality condition for subgroups is extremal (theo-

Chernikov's theorem ⁽⁹⁾, we may restrict ourselves to considering the case when the group G has involutions.

We shall show that a Sylow 2-subgroup S of G cannot be infinite. Let S' be a maximal complete abelian subgroup of S . As is easy to see, the subgroup S' and its normalizer $N_G(S')$ in G satisfy all the conditions of the theorem formulated above, and therefore, for the group G , one of assertions 1)–4) of this theorem holds. Since G is an infinite simple group, it is isomorphic to a group of type $PGL(2, K)$ over a field K of odd characteristic. However, an infinite group of type $PGL(2, K)$ does not satisfy the minimality condition for subgroups. The contradiction obtained shows that the subgroup S cannot be infinite.

Thus we have proved:

The minimality problem for subgroups in a locally finite group is sufficiently solved for the case when its Sylow 2-subgroup S is finite and $S \neq 1$.

Received
1 IX 1965

REFERENCES

- ¹ V. P. Shunkov, DAN, 163, No. 4 (1965).
- ² A. G. Kurosh, S. N. Chernikov, UMN, 2, No. 3, 18 (1947).
- ³ S. N. Chernikov, UMN, 14, no. 5, 45 (1959).
- ⁴ M. Hall, *Theory of Groups*, IL, 1962, p. 227.
- ⁵ Dickson, *Collected Works on Mathematics*, 7, 3, 57 (1963).
- ⁶ G. Higman, J. London Math. Soc., 32, 321 (1957).
- ⁷ W. Feit, Am. J. Math., 82, No. 2, 281 (1960).
- ⁸ W. Feit, J. Thompson, Pacific J. Math., 13, No. 3, 775 (1963).
- ⁹ A. G. Kurosh, *Theory of Groups*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.