

INVESTIGATION OF THE CAUCHY PROBLEM BY THE METHOD OF WEAK APPROXIMATION

N. N. Yanenko, G. V. Demidov

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.62811>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.919

MATHEMATICS

N. N. YANENKO, G. V. DEMIDOV

INVESTIGATION OF THE CAUCHY PROBLEM BY THE METHOD OF WEAK APPROXIMATION

(Presented by Academician L. V. Kantorovich on 16 VII 1965)

In the paper ⁽¹⁾ the question of convergence of the method of fractional steps in differential form was investigated in solving a well-posed Cauchy problem in a Banach space. In the present paper this question is considered without the assumption of well-posedness of the original Cauchy problem, and it is shown that its well-posedness is a consequence of the uniform well-posedness of a certain auxiliary Cauchy problem. The proposed method of investigation is based on the idea of weak approximation of differential operators.

Definition 1. A family of functions $F_\tau(x, t)$ **weakly approximates with respect to t** the function $F(x, t)$ for $0 < t < T$ and $x \in \Omega$, $\Omega \subset E^m$, if

$$\int_{t_1}^{t_2} [F_\tau(x, s) - F(x, s)] ds = \delta(x, t_1, t_2, \tau) \quad (1)$$

and $\|\delta\| \rightarrow 0$ as $\tau \rightarrow 0$ for any fixed admissible t_1, t_2 . A family of linear differential operators $L_\tau(x, t)$ weakly approximates with respect to t the operator $L(x, t)$, if weak approximation holds for the coefficients.

The integral in (1) should be understood in the Riemann sense, except for the case of the space $L_q(\Omega)$, where the Lebesgue integral is also used.

We investigate the well-posedness of the formulation of the Cauchy problem

$$\partial u / \partial t = Lu, \quad 0 < t < T; \quad u|_{t=0} = u_0, \quad u_0 \in B \quad (I)$$

in the Banach space B . With respect to the space B we assume: a) B is a space of functions of m real variables $x_1, x_2, \dots, x_m$, which may depend parametrically on t ; b) in B the operations of differentiation with respect to the spatial variables $\partial/\partial x_k$ and with respect to time $\partial/\partial t$ are defined and are closed with respect to strong convergence; c) sufficiently smooth functions are dense in B .

Let the operator L be representable as a sum

$$L = L_1 + L_2 + \dots + L_p$$

of linear operators of the form

$$L_i = \sum_{k_1, \dots, k_m} a_{k_1 \dots k_m}^i(x_1, \dots, x_m, t) \frac{\partial^{k_1 + \dots + k_m}}{\partial x_1^{k_1} \dots \partial x_m^{k_m}}, \quad (2)$$

where $a_{k_1 \dots k_m}^i$ are real functions, bounded and continuous in t in the uniform topology. If the function $a(x_1, \dots, x_m, t)$ has all the derivatives entering into the expressions (2), then we shall say that it has derivatives up to order L ; if such a differentiation procedure can be repeated j times, then we shall say that $a(x_1, \dots, x_m, t)$ has derivatives up to order $(L)^j$. Let $a_{k_1 \dots k_m}^{k_1}$ have derivatives with respect to the spatial variables up to order $(L)^j$. Then, formally differentiating the equation and the initial data of problem (I), we obtain the problem

$$\partial u^j / \partial t = L^j u^j, \quad 0 < t < T; \quad u^j|_{t=0} = u_0^j, \quad (I^j)$$

where $L^j = L_1^j + L_2^j + \dots + L_p^j$.

Here u^j denotes the vector function consisting of u and of the derivatives of u with respect to the spatial variables up to order $(L)^j$; L_i^j is the operator naturally corresponding to the operator L_i , if one adheres to the rule

$$\frac{\partial Mv}{\partial x_k} = \frac{\partial M}{\partial x_k} v + M \frac{\partial v}{\partial x_k},$$

where v is a component of the vector u^j ; M is a linear differential operator, and $\partial v / \partial x_k$ is understood as a component of the vector u^j . As the norm of u^j we choose the Euclidean vector norm, denoting by B_j the set of functions in B for which this norm is finite.

Let there be associated with the parameter θ a family of problems

$$\partial u / \partial t = Lu, \quad 0 \leq \theta < t < T; \quad u|_{t=0} = u_0, \quad u_0 \in B. \quad (3)$$

Definition 2. Problem (I) is **uniformly well-posed in B for $0 < t < T$** if: a) problem (3) is uniquely solvable for a set of functions u_0 dense in B , i.e. $u(t) = S(t, \theta)u_0$; b) the transition operator $S(t, \theta)$ has the following properties:

$$S(t_2, \theta) = S(t_2, t_1)S(t_1, \theta), \quad 0 \leq \theta < t_1 < t_2 < T; \quad (4)$$

$$\|S(t_2, t_1)\| \leq e^{\alpha(t_2 - t_1)}, \quad 0 \leq t_1 < t_2 < T; \quad (5)$$

$$\|S(t, \theta)u_0 - u_0\| \xrightarrow[t \rightarrow \theta]{} 0, \quad 0 \leq \theta < t < T. \quad (6)$$

If the operator $S(t, \theta)$ satisfies conditions (4)–(6) in B , we shall also say that it satisfies the conditions of uniform well-posedness. In terms of semigroup theory, an operator $S(t, \theta)$ satisfying conditions (4)–(6) belongs to the class $(C_0)^{(2)}$.

To problem (I) we associate the auxiliary (factorized) problem

$$\partial u_\tau / \partial t = L_\tau u_\tau, \quad 0 < t < T; \quad u_\tau|_{t=0} = u_0, \quad u_0 \in B, \quad (I_\tau)$$

where

$$L_\tau = \sum_{i=1}^p \alpha_i(t, \tau) L_i,$$

$$\alpha_i(t, \tau) = \begin{cases} p, & t \in \left(\left(n + \frac{i-1}{p} \right) \tau, \left(n + \frac{i}{p} \right) \tau \right], \\ 0 & \text{in the remaining cases.} \end{cases}$$

The properties of the solutions of the family of problems (I_τ) are completely determined by the properties of the solutions of the problems

$$\partial u^j / \partial t = p L_i^j u^j, \quad 0 < t < T; \quad (I_i^j)$$

$$u^j|_{t=0} = u_0^j, \quad u_0 \in B_j, \quad j = 0, 1, \dots, \quad u^0 \equiv u, \quad B_0 \equiv B.$$

It is not difficult to see that the transition operator of problem (I_τ) is the product of the transition operators of the problems (I_i^0) ,

$$S_\tau(t + \tau, t) = S_p \left(t + \tau, t + \frac{p-1}{p} \tau \right) \cdots S_1 \left(t + \frac{\tau}{p}, t \right) \quad \text{for } t = n\tau.$$

Theorem 1. *If the problems (I_i^0) , (I_i^1) , (I_i^2) are uniformly well-posed, then $u_\tau(t)$ converges uniformly in t to the function $u(t) = S(t, 0)u_0$ as $\tau \rightarrow 0$, and the transition operator $S(t, 0)$ satisfies the conditions of uniform well-posedness.*

Theorem 2. *If the problems (I_i^0) , (I_i^1) , (I_i^2) , (I_i^3) are uniformly well posed, then $u_\tau(t)$, together with its derivatives with respect to x_k up to order L , converges*

strongly, uniformly in t , to the function $u(t)$ as $\tau \rightarrow 0$. The limiting function $u(t)$ has a derivative with respect to t and is a solution of problem (I).

In the terms of semigroup theory, Theorems 1 and 2 give sufficient conditions for the closure of a certain restriction of the sum of the infinitesimal generators of semigroups of class (C_0) also to be an infinitesimal generator of a semigroup of class (C_0) . The first results on the identification of the class of operators that are infinitesimal generators of semigroups are due to Hille ⁽³⁾ and Yosida ⁽⁴⁾. The further development of this theory and its application to the study of the abstract Cauchy problem are set forth in the monograph ⁽²⁾. Semigroups with a t -dependent infinitesimal generator and the Cauchy problem associated with them were studied by P. Lax and R. Richtmyer ⁽⁵⁾. A survey of research on the abstract Cauchy problem is given in ⁽⁶⁾.

Let B_θ^1 be the space with norm

$$\|f\|_{B_\theta^1} = \int_\theta^T \|f\|_B dt.$$

To the inhomogeneous problem

$$\partial u / \partial t = Lu + f, \quad 0 < t < T; \quad u|_{t=0} = u_0, \quad u_0 \in B; \quad f \in B^1(I)$$

and to the parameter θ we associate the family of inhomogeneous problems

$$\partial u / \partial t = Lu + f, \quad 0 \leq \theta < t < T; \quad u|_{t=\theta} = u_0, \quad u_0 \in B; \quad f \in B_\theta^1. \quad (7)$$

If, for given u_0 and f , problem (7) is solvable, we introduce the notation

$$v(t) = u(t) - S(t, \theta)u_0,$$

where $S(t, \theta)$ is the evolution operator of problem (3).

Definition 3. Problem (I) is **well posed with respect to the right-hand side** for $0 < t < T$ if $v(t)$ depends continuously on f .

Let $u(t)$ be a solution of problem (I). The following holds.

Theorem 3. If: a) $u(0) = u_0 \in B_1$; b) $L_i u$ are uniformly continuous in t ; c) the problems (I_i^0) , (I_i^1) are uniformly well posed; d) the problems (I^0) are well posed with respect to the right-hand side, then $u(t)$ is the unique solution of problem (I) satisfying a) and b), and $u_\tau(t)$ converges to $u(t)$ uniformly in t .

In the case when $B \equiv L_q(\Omega)$, $q > 1$, $\Omega \subset E^m$, we shall assume that $u^1(t)$ includes at least all first derivatives of $u(t)$ with respect to the spatial variables. We shall say that $u(t)$ is a **smooth function** if it has all derivatives entering into the equation of problem (I), belonging to $L_q(\Omega^1)$, $\Omega^1 = \Omega \times (0, T)$.

Theorem 4. If: a) the problems (I_i^0) , (I_i^1) are uniformly well posed; b) the problems (I^0) are well posed with respect to the right-hand side, then $u_\tau(t)$

converges weakly as $\tau \rightarrow 0$, uniformly in t , and every smooth limiting function $u(t)$ is a solution of problem (I); in the case $u_0^1 \in L_q(\Omega)$, $u(t)$ is the smooth and unique solution of problem (I).

Finally, if $m = p$, $q = 2$, $L_i = A_i(x, t)\partial/\partial x_i$, where A_i are symmetric matrices, continuous in Ω^1 together with their first derivatives with respect to the spatial variables, and Ω^1 is the usual cone of dependence, then the following holds.

Theorem 5. *The problems (I), (I_i^0) are uniformly well posed. The function $u_\tau(t)$ converges uniformly in t , as $\tau \rightarrow 0$, to the solution of problem (I).*

Computing Center

Siberian Branch of the Academy of Sciences of the USSR

Received

10 VII 1965

REFERENCES

1. N. N. Yanenko, *Siberian Mathematical Journal*, **5**, No. 6, 1431 (1964).
2. E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, IL, 1962.
3. E. Hille, *Functional Analysis and Semigroups*, IL, 1951.
4. K. Yosida, *J. Math. Soc. Japan*, **1**, 15 (1948).
5. P. D. Lax, R. D. Richtmyer, *Comm. Pure and Appl. Math.*, **9**, 267 (1956).
6. M. A. Krasnosel'skii, S. G. Krein, *Proceedings of the IV All-Union Mathematical Congress*, 2, L., 1964, p. 292.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.