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Abstract

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MATHEMATICS

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ON THE NORMAL SOLVABILITY OF THE DIRICHLET PROBLEM FOR A SYSTEM OF ELLIPTIC TYPE OF A. V. BITSADZE

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Among elliptic boundary-value problems for which the Noether property is violated, the most typical is the Dirichlet problem for the system of A. V. Bitsadze

$$\begin{aligned}\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} - 2 \frac{\partial^2 u_2}{\partial x \partial y} &= F_1(x, y), \\ \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} + 2 \frac{\partial^2 u_1}{\partial x \partial y} &= F_2(x, y); \end{aligned} \quad (1)$$

$$u_1|_{\Gamma} = f_1(s), \quad u_2|_{\Gamma} = f_2(s). \quad (2)$$

In the work ⁽¹⁾ it was shown that the homogeneous problem $(1^0)-(2^0)$ ($F_1 \equiv F_2 \equiv 0$, $f_1 \equiv f_2 \equiv 0$) has in a disk an infinite number of linearly independent solutions, while the nonhomogeneous problem for the same domain is normally solvable in the sense of Hausdorff ⁽³⁾.

The aim of the present work is to study the nature of the violation of the Noether property of problem (1)–(2) for a broad class of noncircular domains.

In complex form, problem (1)–(2) can be written as

$$\partial^2 w / \partial \bar{z}^2 = F(z), \quad (1')$$

$$w|_{\Gamma} = f(s), \quad (2')$$

where $w = u_1 + iu_2$, $F = F_1 + iF_2$, $f = f_1 + if_2$.

It is assumed that the derivative $\partial^2 / \partial \bar{z}^2$ is understood in the sense of S. L. Sobolev, $F(z) \in L_p(\mathcal{D})$, $p > 1$, $\mathcal{D}$ is a simply connected domain with Lyapunov

boundary Γ , and the boundary condition (2') is satisfied almost everywhere, with $f(s) \in L_2(\Gamma)$.

Without loss of generality, one may assume that $F(z) \equiv 0$, i.e.

$$\partial^2 u / \partial \bar{z}^2 = 0; \quad (\text{I})$$

$$u|_{\Gamma} = f(s). \quad (\text{II})$$

This is easily verified by replacing the sought solution by

$$w = u + \frac{1}{\pi^2} \iint_D \frac{d\xi_1 d\eta_1}{t_1 - z} \iint_D \frac{F(\xi, \eta)}{t - t_1} d\xi d\eta.$$

The general solution of equation (I), as is known ⁽²⁾, has the form

$$u(z) = \bar{z}\varphi(z) + \psi(z), \quad (3)$$

where $\varphi(z)$, $\psi(z)$ are arbitrary holomorphic functions in $\mathcal{D}$.

Definition. A solution $u(z)$ of equation (I) will be called **regular** if the functions $\varphi(z)$ and $\psi(z)$ in representation (3) belong to the class $E_2(\mathcal{D})$ ⁽⁴⁾.

We shall call problem D the problem of finding regular solutions in $\mathcal{D}$ of equation (I), satisfying almost everywhere the boundary condition (II).

Below we study both the homogeneous problem D_0 ($f \equiv 0$) and the nonhomogeneous problem D.

Problem (I)–(II) is easily reduced to a Fredholm integral equation of the first kind. Therefore the violation of Noetherianity for this problem and for a Fredholm integral equation of the first kind has the same character.

Let $z(\omega)$ be a holomorphic function in the unit disk S , continuous in the closed domain $S + \gamma$; let $\{\varphi_i(\omega)\}$ be the system of functions determined from the expansion of $z(\omega)$ into a power series near zero

$$z(\omega) = \sum_{k=0}^{i-1} a_k \omega^k + \omega^i \varphi_i(\omega). \quad (4)$$

Lemma 1. In the system $\{\varphi_i(\omega)\}$, either all the functions are linearly independent, or among them there exists only a finite number m of linearly independent functions $\varphi_1(\omega), \varphi_2(\omega), \dots, \varphi_m(\omega)$. The latter case occurs if and only if $z(\omega)$ is a rational function.

Let now $z(\omega)$ be a conformal mapping of the unit disk $|\omega| \leq 1$ onto the domain $\mathcal{D}$.

Theorem 1. Among simply connected domains bounded by analytic curves, the homogeneous problem D_0 has nonzero regular solutions if and only if $z(\omega)$ is a rational function.

This fact, under the assumption that $\varphi(z), \psi(z) \in C^{(1,h)}(\mathcal{D} + \Gamma)$, was noted by N. E. Tovmasyan.

Along with problem (I)–(II), consider the homogeneous adjoint problem

$$\partial^2 v / \partial z^2 = 0; \quad (\text{I}^*)$$

$$v|_{\Gamma} = 0. \quad (\text{II}^*)$$

Starting from Green's formula

$$\iint_{\mathcal{D}} \left[\bar{v} \frac{\partial^2 u}{\partial \bar{z}^2} - u \frac{\partial^2 \bar{v}}{\partial z^2} \right] dx dy = \frac{1}{2i} \int_{\Gamma} \left(\bar{v} \frac{\partial u}{\partial \bar{z}} - u \frac{\partial \bar{v}}{\partial z} \right) dz$$

for a solution $u(z)$ of problem (I)–(II) and for any solution $v(z)$ of the homogeneous adjoint problem (I)–(II), we have

$$\int_{\Gamma} f \frac{\partial \bar{v}}{\partial z} dz = 0. \quad (5)$$

Condition (5) is necessary for the solvability of problem (I)–(II). If condition (5) is also sufficient for the solvability of problem (I)–(II), then the latter problem is called **Hausdorff**. As a result of work [5] and Lemma 1 it follows

Theorem 2. For a simply connected domain bounded by a Lyapunov curve, problem (I)–(II) is Hausdorff if and only if $z(\omega)$ is a rational function.

From Theorems 1 and 2 it follows

Theorem 3. For a simply connected domain bounded by a Lyapunov curve, problem (I)–(II) is not Noetherian.

Let $\varphi_i(\omega)$ be the system of functions determined from (4), where $z(\omega)$ is the indicated conformal mapping, and let $\omega(z)$ be the function inverse to $z(\omega)$. Denote by $\Omega^+(s)$ the boundary value on Γ of any function of the class $E_2(\mathcal{D})$.

Theorem 4. In the case of the Hausdorff property of problem (I)–(II), for its solvability it is necessary and sufficient that the right-hand side $f(s)$ in condition II have the form

$$f(s) = \Omega^+(s) + \sum_{i=1}^m c_i \overline{\omega(s)} \overline{\varphi_i(\omega(s))},$$

where c_i are arbitrary constants.

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Note: Figure translations are in progress. See original paper for figures.

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