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N. U. Arakelian

1966

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Abstract

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UDC 517.535.4

MATHEMATICS

N. U. ARAKELYAN

ENTIRE FUNCTIONS OF FINITE ORDER WITH AN INFINITE SET OF DEFICIENT VALUES

(Presented by Academician M. V. Keldysh on 11 VII 1966)

A well-known conjecture of R. Nevanlinna states ^(1,2) that an entire function of finite order ρ has only a finite number (not more than $[2\rho] + 1$) of deficient values. For the case $\rho \leq 1/2$ this conjecture has been confirmed. Edrei and Fuchs showed, in particular ⁽³⁾, that an entire function of order $\rho \leq 1/2$ cannot have finite deficient values. It turns out that in the case $\rho > 1/2$ R. Nevanlinna's conjecture is false. Moreover, the following is true.

Theorem. For an arbitrary sequence of complex numbers $\{a_k\}_1^\infty$ and for any $\rho > 1/2$, there exists an entire function of order ρ and of normal type for which the numbers a_k , $k = 1, 2, \dots$, are deficient.

1°. The proof of this theorem is based on the following two lemmas from approximation theory.

Lemma 1. Let L denote a rectifiable curve joining the points a and b . Then for arbitrary numbers $\varepsilon > 0$ and $d > 0$ there exists a polynomial $P(z)$ such that

$$\left| \frac{1}{a-z} - P\left(\frac{1}{z-b}\right) \right| < \varepsilon \quad (1)$$

for all points z lying outside the d -neighborhood of the curve L , and

$$\left| P\left(\frac{1}{z-b}\right) \right| < \exp \left[(1 + |\ln \varepsilon d|) \exp \frac{5 \text{ length } L}{d} \right] \quad (2)$$

outside the circle $|z-b| < d$.

Lemma 2. Let $1/2 < \rho \leq 1$, $0 < \alpha < \frac{\pi}{\rho}(\rho - 1/2)$, and let the function $F(z)$ be analytic in the angle $|\arg z| \leq \beta/2$ ($\beta > \alpha$) and satisfy $|F(z)| < \exp[\text{const}(|z| + 1)^\rho]$. Then for any $\varepsilon > 0$ there exists an entire function $G(z)$ of order ρ and of normal type satisfying the inequality $|F(z) - G(z)| < \varepsilon \exp(-|z|^\rho)$, $|\arg z| \leq \alpha/2$.

We note that Lemma 1 is a special case of an important lemma of M. V. Keldysh⁽⁴⁾, while Lemma 2 is an immediate consequence of a theorem of M. V. Keldysh⁽⁵⁾ on approximation in an angle by entire functions of restricted growth.

2°. In proving the theorem it is sufficient to restrict ourselves to the case when $1/2 < \rho \leq 1$. Indeed, if $\rho > 1$, choose a natural number N such that $\rho \leq N < 2\rho$. Having constructed an entire function $G(z)$ of order ρ/N with deficient values $\{a_k\}_1^\infty$, we may then consider the entire function $G(z^N)$ of order ρ with the same defects.

3°. Now take an arbitrary number α , $0 < \alpha < \frac{\pi}{\rho}(\rho - 1/2)$, and a sequence $\{\theta_k\}_{-\infty}^{+\infty}$ such that $\theta_{-k} = -\theta_k$, $\theta_k \uparrow +\infty$ as $k \uparrow +\infty$. Denote by $\Delta_k(\varepsilon)$ the angle $|\arg z - \theta_k| \leq \varepsilon \alpha_k$, where $\alpha_k = \min(\theta_{k+1} - \theta_k, \theta_k - \theta_{k-1})$, by $\sigma_n(\delta)$ ($\delta \geq 0$) the annulus $(1 - \delta)2^n \leq |z| \leq (1 + \delta)2^{n+1}$, and put $E_{k,n}(\varepsilon, \delta) = \Delta_k(\varepsilon) \cap \sigma_n(\delta)$. Let now

now $\{n_k\}_1^\infty$ is some sequence of even natural numbers, $n_k \uparrow +\infty$ as $k \uparrow +\infty$. Setting $n_{-k} = n_k + 1$, consider the closed sets

$$E(\varepsilon, \delta) = \bigcup_{|k|=1}^{+\infty} \bigcup_{m=0}^{+\infty} E_{k, n_k+2m}(\varepsilon, \delta),$$

$$E_0 = E(1/4, 0), \quad E_1 = E(1/2, 1/8), \quad E_2 = E(3/4, 1/4),$$

so that $E_0 \subset E_1 \subset E_2$. Setting further

$$\varepsilon_k = \exp(-2^{12} a_k^{-1}), \quad k = \pm 1, \pm 2, \dots,$$

we shall assume that the n_k increase so rapidly that

$$\prod_{|k|=1}^{+\infty} \prod_{n=n_k}^{+\infty} \frac{1 + \delta_{k,n}}{1 - \delta_{k,n}} < 16/15, \quad \text{where } \delta_{k,n} = \exp(-\varepsilon_k 2^{n\rho}), \quad (3)$$

$$\max(a_k^{-1}, 2^{n_k}, |a_k|) < \exp(\varepsilon_k 2^{n_k \rho - 1}), \quad k = 1, 2, \dots, \quad (4)$$

$$\int_{\partial E_1} |\zeta^{-2} d\zeta| < 1. \quad (5)$$

4°. We now denote by $\gamma_{k,n}^i$ ($i = 1, 2$; $k = \pm 1, \pm 2, \dots$; $n = n_k + 2m$; $m = 1, 2, \dots$) the boundary of the set $E_i \cap E_{k,n}(3/4, 1/4)$. The curve

$$l_{k,n}^i : z = (1 + 2^{i-4})2^{n+1} e^{i\theta}, \quad (-1)^{n_k} \theta_n \leq (-1)^{n_k} \theta \leq \pi,$$

joins the curve $\gamma_{k,n}^i$ with the point $-(1 + 2^{i-4})2^{n+1}$. By $D_{k,n}^i$ we denote the d -neighborhood of the curve $\gamma_{k,n}^i \cup l_{k,n}^i$, where $d = a_k 2^{n-4}$. It is easy to verify that $E_{i-1} \subset CD_{k,n}^i$, and that the length $\gamma_{k,n}^i \cup l_{k,n}^i < 2^{n+4}$. Take an arbitrary point $\zeta \in \gamma_{k,n}^i$ and apply Lemma 1 to the domain $D_{k,n}^i$, putting

$$a = \zeta, \quad b = -(1 + 2^{i-4})2^{n+1}, \quad \varepsilon = 4^{-n-2} \exp(-5\varepsilon_k 2^{n\rho}).$$

We may suppose that the function

$$Q_i(\zeta, z) \equiv P(1/(z - b))$$

is analytic in z in the half-plane $\operatorname{Re} z > -1$, is defined and piecewise constant in ζ (uniformly with respect to z) on the set ∂E_i , and satisfies, by virtue of (1), (2), and (4), the inequalities

$$|Q_i(\zeta, z) - 1/(\zeta - z)| < 4^{-n-2} \exp(-5\varepsilon_k 2^{n\rho}) \quad \text{for } \zeta \in \gamma_{k,n}^i, \quad z \in CD_{n,k}^i, \quad (6)$$

$$|Q_i(\zeta, z)| < \exp(\varepsilon_k^{1/2} 2^{n\rho+4}) \quad \text{for } \zeta \in \gamma_{k,n}^i, \quad \operatorname{Re} z > -1. \quad (7)$$

5°. We now define two functions $\varphi(z)$ and $\psi(z)$, analytic on the set E_2 , by putting

$$\varphi(z) = a_{|k|}, \quad \psi(z) = \exp(-\varepsilon_k z^\rho), \quad \text{when } z \in \Delta_k(3/4), \quad k = \pm 1, \pm 2, \dots \quad (8)$$

Lemma 3. There exists a function $\omega(z)$, analytic in the half-plane $\operatorname{Re} z > -1$, such that

$$1 < |\omega(z)/\psi(z)| < 2 \quad \text{for } z \in E_1, \quad (9)$$

$$|\omega(z)| < \exp[(|z| + 1)^\rho] \quad \text{for } \operatorname{Re} z > -1. \quad (10)$$

Proof. Consider the function

$$\omega_{k,n}(z) = 1 + \frac{1}{2\pi i} \int_{\gamma_{k,n}^2} [\psi(\zeta) - 1] Q_2(\zeta, z) d\zeta, \quad (11)$$

analytic in the half-plane $\operatorname{Re} z > -1$. Observing now that, by the Cauchy formula,

$$\psi_{k,n}(z) = 1 + \frac{1}{2\pi i} \int_{\gamma_{k,n}^2} \frac{\psi(\zeta) - 1}{\zeta - z} d\zeta = \begin{cases} \psi(z), & \text{for } z \in E_{k,n}(1/2, 1/8), \\ 1, & \text{for } z \in CE_{k,n}(3/4, 1/4), \end{cases}$$

and taking into account that the length of $\gamma_{k,n}^2 < 2^{n+3}$, from (6) and (11) we obtain

$$|\omega_{k,n}(z) - \psi_{k,n}(z)| < 1/4 \exp(-5\varepsilon_k 2^{n\rho}) \quad \text{for } z \in CD_{k,n}^i,$$

whence

$$3/4 < |\omega_{k,n}(z)/\psi(z)| < 5/4 \quad \text{for } z \in E_{k,n}(1/2, 1/8), \quad (12)$$

$$|\omega_{k,n}(z) - 1| < \exp(-\varepsilon_k 2^{n\rho}) \quad \text{for } z \in CD_{k,n}^2 \cap CE_{k,n}(3/4, 1/4). \quad (13)$$

Inequality (13) holds, in particular, for $z \in E_1 \setminus E_{k,n}(1/2, 1/8)$.

The growth of the function $\omega_{k,n}(z)$ is bounded, by virtue of (7), (11), and (4), by the inequality

$$|\omega_{k,n}(z)| < \exp\left(\varepsilon_k^{1/2} \cdot 2^{n\rho+5}\right), \quad \operatorname{Re} z > -1. \quad (14)$$

Now define the desired function $\omega(z)$ by the formula

$$\omega(z) = 3/2 \prod_{|k|=1}^{+\infty} \prod_{m=0}^{+\infty} \omega_{k,n_k+2m}(z).$$

The convergence of this product follows from (13) and (3). To obtain estimate (9), it suffices to use (12), (13), and (3). Finally, to estimate the growth of $\omega(z)$, take an arbitrary number z , $\operatorname{Re} z > -1$. Let $z \in \sigma_N(0)$. Then $z \in CD_{k,n}^2 \cap CE_{k,n}(3/4, 1/4)$ for all n , except, possibly, for the cases $n = N-1, N, N+1$. Taking (13), (14), and (3) into account, we obtain:

$$|\omega(z)| < 2 \prod_{|k|=1}^{+\infty} \prod_{n=N-1}^{N+1} |\omega_{k,n}(z)| \exp(2^{N\rho}) < \exp[(|z| + 1)^\rho].$$

Lemma 4. *There exists a function $F(z)$, analytic in the angle $|\arg z| \leq \pi/2$, such that*

$$|\varphi(z) - F(z)| < 1/2 |\psi(z)| \quad \text{for } z \in E_0,$$

$$|F(z)| < \exp[\text{const} \cdot (|z| + 1)^\rho] \quad \text{for } |\arg z| \leq \pi/2.$$

Proof. We note that from (6) the estimate follows

$$|\theta_1(\xi, z) - 1/(\xi - z)| < |\psi(\xi)|/|\xi|^2, \quad \xi \in \partial E_1, \quad (15)$$

valid for $z \in E_0$, and also in the disk $|z| \leq 1/4|\xi|$. The growth of the function $Q_1(\xi, z)$ is bounded, by virtue of estimate (7), by the inequality

$$|Q_1(\xi, z)| < \exp(|\xi|^\rho) \quad \text{for } \xi \in \partial E_1, \text{ Re } z > -1. \quad (16)$$

Now denote

$$\Gamma_n = \bigcup_{\substack{(k,m) \\ n_k + 2m \leq n}} \gamma_{k, n_k + 2m} \quad \text{for } n \geq n_1$$

(so that $\partial E_1 = \bigcup_{n=n_1}^{+\infty} \Gamma_n$) and consider the sequence of functions analytic for $\text{Re } z > -1$,

$$H_n(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\varphi(\xi)}{\omega(\xi)} Q_1(\xi, z) d\xi, \quad (17)$$

where $\omega(z)$ is the function constructed in Lemma 3. From inequalities (4) and (10) (taking account of (8)) it follows that

$$|\varphi(\xi)/\omega(\xi)| < 1/|\psi(\xi)|^2, \quad \xi \in E_1. \quad (18)$$

Take a natural number N , and let $|z| \leq 2^N$, $m > n > N + 1$. Then

$$0 \equiv \frac{1}{2\pi i} \int_{\Gamma_m \setminus \Gamma_n} \frac{\varphi(\xi)}{\omega(\xi)} \frac{d\xi}{\xi - z},$$

whence, taking (15), (17), (18), and (5) into account, we obtain:

$$|H_m(z) - H_n(z)| < \frac{1}{2\pi} \int_{\partial E_1 \setminus \Gamma_n} |\xi^{-2} d\xi| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (19)$$

It follows that the sequence $H_n(z)$ converges uniformly in every disk to some function $H(z)$ analytic for $\text{Re } z > -1$. Similarly, taking into account the formula

$$\frac{\varphi(z)}{\omega(z)} = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\varphi(\xi)}{\omega(\xi)} \frac{d\xi}{\xi - z}, \quad z \in E_0, \quad |z| \leq 2^n,$$

for an arbitrary point $z \in E_0$ we have the estimate

$$\left| H(z) - \frac{\varphi(z)}{\omega(z)} \right| = \lim_{n \rightarrow \infty} \left| H_n(z) - \frac{\psi(z)}{\omega(z)} \right| \leq \frac{1}{2\pi} \int_{\partial E_1} |\xi^{-2} d\xi| < 1/4. \quad (20)$$

Let us now estimate the growth of the function $H(z)$. Let $z \in \sigma_{N-1}(0)$. If $\zeta \in \Gamma_{N+2}$, then $|\zeta| \leq 18|z|$. Taking now (19), (17), (18), (16), and (15) into account, we obtain

$$\begin{aligned} |H(z)| &< 1 + |H_{N+2}(z)| < 1 + \max_{\zeta \in \Gamma_{N+2}} \left| \frac{\zeta}{\psi(\zeta)} \right|^2 \cdot |Q_1(\zeta, z)| < \\ &< 1 + \max_{|\zeta| \leq 18|z|} |\zeta|^2 \exp(2|\zeta|^\rho) < \exp[\text{const} \cdot (|z| + 1)^\rho]. \end{aligned} \quad (21)$$

By virtue of (20), (9), and (21), the function $F(z) = H(z)\omega(z)$ satisfies all the conditions of Lemma 4.

6°. From Lemmas 4 and 2 there follows the existence of an entire function $G(z)$ of order ρ and of normal type, satisfying the inequality

$$|G(z) - \varphi(z)| < |\psi(z)|, \quad z \in E_0. \quad (22)$$

The function $G(z)$ is the required function with an infinite set of deficient values $\{a_k\}_1^\infty$. Indeed, by virtue of the definition of the set E_0 and of the function $\varphi(z)$, for any $k = 1, 2, \dots$, when $r \geq 2^{n_k}$, there exists an arc $\lambda_{k,r}$ of the circumference $|z| = r$, $\lambda_{k,r} \subset E_0$, such that $\varphi(z) = a_k$ for $z \in \lambda_{k,r}$, and the length $\lambda_{k,r} = \frac{\alpha_k}{2}r$. Then from (22) we obtain that

$$|G(z) - a_k| < |\psi(z)| < \exp\left(-\frac{\varepsilon_k}{2}r^\rho\right), \quad z \in \lambda_{k,r}, \quad r \geq 2^{n_k}.$$

Hence, in the generally accepted notation, we have

$$m(r, a_k, G) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ \frac{1}{|G(re^{i\theta}) - a_k|} d\theta > \frac{\alpha_k \varepsilon_k}{8\pi} r^\rho > \varepsilon_k^2 r^\rho \quad \text{for } r \geq 2^{n_k},$$

and since $T(r, G) \leq \ln^+ M(r, G) < \sigma r^\rho$ for $r \geq 1$, we finally obtain

$$\delta(a_k, G) = \lim_{r \rightarrow \infty} [m(r, a_k, G)/T(r, G)] \geq \varepsilon_k^2/\sigma > 0.$$

The theorem is completely proved.

Remark. Analysis of the example constructed above gives some grounds to suppose that, for an arbitrary entire function $G(z)$ of finite order, the condition

$$\sum_{0 < \delta(a, G) < 1} \frac{1}{\ln \delta^{-1}(a, G)} < +\infty$$

is satisfied.

This condition cannot be further strengthened, since in our example $1/\ln \delta^{-1}(a_k, G) > \text{const} \cdot \alpha_k > 0$, where of the numbers a_k it is required only that the series $\sum_{k=1}^{\infty} \alpha_k$ converge.

Institute of Mathematics and Mechanics
Academy of Sciences of the Armenian SSR

Received
8 VII 1966

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