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Abstract

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MATHEMATICS

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ON THE CAUCHY PROBLEM FOR SECOND-ORDER DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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1. Abstract parabolic equations of second order

Consider the problem

$$u''(t) + Au'(t) + Bu(t) = f(t, u(t), u'(t)); \quad u(0) = u_0, \quad u'(0) = u_1, \quad (1)$$

in a Banach space E . We shall call a function $u(t)$ a solution of problem (1) if, for all t in $[0, T]$, it satisfies equation (1) and the functions $u''(t)$, $Au'(t)$, and $Bu(t)$ are continuous on $[0, T]$.

In this section the following condition is imposed everywhere on the operator A :

(α) Let the closed linear operator A have everywhere dense domain of definition $D(A)$, and let the inequality

$$\|R(\sigma + i\tau, -A)\| \leq C(\sigma + 1 + |\tau|^\alpha)^{-1}$$

hold for all $\sigma > -1$ and for some $\alpha \in (0, 1]$.

If the operator A satisfies condition (α), then $-A$ generates ¹ an infinitely differentiable semigroup $T(t)$ of class A . This class of semigroups is usually denoted by $(A)_\infty$. If $1/2 < \alpha < 1$, then $T(t)$ belongs to the class $(1, A)_\infty$; if $\alpha = 1$, then $T(t)$ is an analytic semigroup. For operators satisfying condition (α), fractional powers are defined ².

In ^{3,4}, the solvability of problem (1) was investigated for $\alpha = 1$. Using the results and the method of paper ⁵, we investigate the solvability of problem (1)

for all $\alpha \in (1/2, 1]$, and for $\alpha = 1$ the results obtained in this note coincide with the results of papers ^{3,4}.

Lemma 1. Let the operator A satisfy condition (α) . Then the semigroup $T(t)$ satisfies the estimate

$$\|A^\beta T(t)\| \leq C(\beta)t^{(\alpha-\beta-1)/\alpha}$$

for all $\beta \geq 0$, where the function $C(\beta)$ is locally bounded.

Lemma 2. Let the operator A satisfy condition (α) . If the linear operator B is such that, for some $\beta < \alpha$, the operator $BA^{-\beta}$ is bounded, then there exists $k > 0$ for which the inequality

$$\|R(\sigma + i\tau, -A + B - kI)\| \leq C(\sigma + 1 + |\tau|^\alpha)^{-1}$$

holds for all $\sigma > -1$.

Theorem 1. Suppose the following conditions are fulfilled:

1°. The operator A satisfies condition (α) for some $\alpha \in (1/2, 1]$.

2°. The operator $BA^{-1-\beta}$ is bounded for some $\beta < \alpha$, the operator BA^{-1} is closed, and for some ω_0 the inequality

$$\|R(\lambda, -BA^{-1})\| \leq C(|\lambda| + 1)^{-1} \quad (2)$$

holds for all λ with $\operatorname{Re} \lambda \geq \omega_0$.

3°. $u_0 \in D(B) \cap D(AB)$, $u_1 \in D(A^2) \cap D(ABA^{-1})$, $f(0, u_0, u_1) \in D(A)$.

4°. For some $R > 0$ the operator $f(t, A^{-1}v_1, v_2)$ on $[0, T] \times S(Au_0, R) \times S(u_1, R)$ has partial derivatives $f'_t(t, A^{-1}v_1, v_2)$ and $f'_{v_i}(t, A^{-1}v_1, v_2)$, $i = 1, 2$, that are continuous in the aggregate of variables (the latter in the sense of Fréchet), satisfying with respect to v_i a Lipschitz condition ($f'_t(t, A^{-1}v_1, v_2)$ —in the norm of the space E , $f'_{v_i}(t, A^{-1}v_1, v_2)$ —in the norm of the space of linear operators over E).

Then problem (1) has a unique solution, defined on some interval $[0, t_0] \subset [0, T]$, which can be found by the method of successive approximations.

Proof. By the substitution $v_1(t) = Au(t) + u'(t)$, $v_2(t) = u'(t)$, problem (1) is reduced to the equivalent problem

$$v'(t) + \mathbf{A}v(t) = F(t, v(t)), \quad v(0) = v_0,$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad F(t, v) = \begin{pmatrix} f(t, A^{-1}(v_1 - v_2), v_1) \\ f(t, A^{-1}(v_1 - v_2), v_2) \end{pmatrix}, \quad v_0 = \begin{pmatrix} Au_0 + u_1 \\ u_1 \end{pmatrix},$$

$$\mathbf{A} = \begin{pmatrix} BA^{-1} & -BA^{-1} \\ BA^{-1} & A - BA^{-1} \end{pmatrix} = \begin{pmatrix} BA^{-1} + \omega + I & 0 \\ BA^{-1} & A \end{pmatrix} + \begin{pmatrix} -\omega - I & -BA^{-1} \\ 0 & -BA^{-1} \end{pmatrix} = \mathbf{A}_1 + \mathbf{A}_2,$$

$$\omega = \min\{0, \omega_0\}.$$

It is not difficult to verify that $R(\lambda, -A_1)$ is determined by the formula

$$R(\lambda, -A_1) = \begin{pmatrix} R(\lambda, -BA^{-1} - \omega - I) & 0 \\ R(\lambda, -A)BA^{-1}R(\lambda, -BA^{-1} - \omega - I) & R(\lambda, -A) \end{pmatrix}.$$

Then from 1° and 2° we obtain the estimate

$$\|R(\lambda, -A_1)\| \leq C(\sigma + 1 + |\tau|^\alpha)^{-1}.$$

Next it is established that, for $\alpha_0 > \beta$, the operator $\mathbf{A}_2\mathbf{A}_1^{-\alpha_0}$ is bounded. Hence, on the basis of the lemmas and certain results from (5), the theorem is proved.

If one considers the equation with a small parameter

$$\varepsilon u''(t) + Au'(t) + Bu(t) = f(t, u(t), u'(t)); \quad u(0) = u_0, \quad u'(0) = u_1 \quad (3)$$

for sufficiently small $\varepsilon > 0$, then in Theorem 1 the condition of boundedness of the operator $BA^{-1-\beta}$ for some $\beta < \alpha$ is replaced by the weaker condition of boundedness of the operator $BA^{-1-\alpha}$.

If there is a linear estimate for the nonlinear operator $f(t, A^{-1}v_1, v_2)$, solvability of problem (1) (or (3)) on the whole interval $[0, T]$ is proved. If equation (1) (or (3)) is linear, then condition 3° in the corresponding theorem is replaced by the weak condition 3':

$$3_1^\circ. \quad u_0 \in D(A) \cap D(B), \quad u_1 \in D(A).$$

In this case the second derivative of the solution found, as well as the function $Au'(t)$, are continuous only on $(0, T]$.

2. Abstract hyperbolic equations of second order.

Consider the problem

$$u''(t) + A(t)u(t) = f(t, u(t), u'(t)); \quad u(0) = u_0, \quad u'(0) = u_1. \quad (4)$$

By a solution of problem (4) we shall mean a function $u(t)$, if for each t in $[0, T]$ it satisfies equation (4) and the functions $u''(t)$, $A(t)u(t)$, $A^{1/2}(t)u'(t)$ are continuous on $[0, T]$. By $A^{1/2}(t)$ here and below we understand any square root of the operator $A(t)$, i.e. $A^{1/2}(t)A^{1/2}(t) = A(t)$.

In this section problem (4) is studied in a Banach space under assumptions that make it possible to include hyperbolic equations. The study of problem (4) in a Banach (and not Hilbert) space makes it possible, in applications, to consider a much broader class of quasilinear partial differential equations of hyperbolic type. Analogous results for special cases of equation (4) were obtained in (1, 6).

Theorem 2. Suppose that the following conditions are fulfilled:

1°. The operator $A^{1/2}(t)$ has an everywhere dense domain of definition independent of t , $D(A^{1/2}(t)) = D(A^{1/2})$, and for some $\omega \geq 0$ the estimate

$$\|R(i\lambda, A^{1/2}(t))\| \leq \frac{1}{|\lambda| - \omega}$$

holds for all real λ satisfying the condition $|\lambda| > \omega$.

*

Condition 1° is a necessary and sufficient condition for $iA^{1/2}(t)$ to generate a group of bounded operators with norm $\|e^{i\tau A^{1/2}(t)}\| \leq e^{\omega|\tau|}$.

2°. The operator-function $A^{1/2}(t)A^{-1/2}(0)$ is twice strongly continuously differentiable on $[0, T]$.

3°. $u_0 \in D(A(0))$, $u_1 \in D(A^{1/2})$.

4°. The operator $f(t, A^{-1/2}(0)v_1, v_2)$ on $[0, T] \times S(A^{1/2}(0)u_0, R) \times S(u_1, R)$, where R is some number, has partial derivatives continuous in the aggregate of the variables $f'_t(t, A^{-1/2}(0)v_1, v_2)$, $f'_{v_i}(t, A^{-1/2}(0)v_1, v_2)$, $i = 1, 2$ (the latter derivatives being understood in the Fréchet sense), satisfying, with respect to v_i , $i = 1, 2$, a Lipschitz condition.

Then there exists a unique solution of problem (4) on some interval $[0, t_0] \subset [0, T]$, which can be found by the method of successive approximations.

Proof. By means of the substitution $v_1(t) = \frac{1}{2}[iA^{1/2}(t)u(t) + u'(t)]$, $v_2(t) = \frac{1}{2}[-iA^{1/2}(t)u(t) + u'(t)]$, problem (4) is reduced to the equivalent Cauchy problem for a first-order equation

$$v'(t) + \mathbf{A}(t)v(t) = F(t, v(t)), \quad v(0) = v_0, \quad (5)$$

where

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad v_0 = \begin{pmatrix} \frac{1}{2}[iA^{1/2}(0)u_0 + u_1] \\ \frac{1}{2}[-iA^{1/2}(0)u_0 + u_1] \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} -iA^{1/2}(t) + \omega I & 0 \\ 0 & +iA^{1/2}(t) + \omega I \end{pmatrix},$$

$$F(t, v) = \begin{pmatrix} \frac{1}{2}[A_t^{1/2}(t)A^{-1/2}(t)(v_1 - v_2) + f(t, -iA^{-1/2}(t)(v_1 - v_2), v_1 + v_2)] + \omega v_1 \\ \frac{1}{2}[-A_t^{1/2}(t)A^{-1/2}(t)(v_1 - v_2) + f(t, -iA^{-1/2}(t)(v_1 - v_2), v_1 + v_2)] + \omega v_2 \end{pmatrix},$$

after which the results of works (7, 8) are applied to problem (5).

Nonlocal existence and uniqueness theorems can be obtained if, for all solutions of equations (4), a priori estimates $\|A^{1/2}(t)u(t)\| \leq C$, $\|u'(t)\| \leq C$ are established. The latter is established, for example, if the operator $f(t, A^{-1/2}(0)v_1, v_2)$ has a linear majorant, i.e. on $[0, T] \times E \times E$ the estimate $\|f(t, A^{-1/2}(0)v_1, v_2)\| \leq C(1 + \|v_1\| + \|v_2\|)$ holds. In a Hilbert space a stronger result is proved, namely, a nonlocal theorem is established under one-sided estimates imposed on the operator $f(t, A^{-1/2}(0)v_1, v_2)$.

Theorem 3. Let the following conditions be fulfilled:

1°. For each t from $[0, T]$, $A(t)$ is a self-adjoint operator, and for all $u \in D(A(t))$ and $t \in [0, T]$ one has

$$(A(t)u, u) \geq \gamma^2(u, u).$$

2°. The operator $A^{1/2}(t)$ has a domain of definition $D(A^{1/2})$ independent of t ; the operator-function $A^{1/2}(t)A^{-1/2}(0)$ is twice strongly continuously differentiable on $[0, T]$.

3°. $u_0 \in D(A(0))$, $u_1 \in D(A^{1/2})$.

4°. The operator $f(t, A^{-1/2}(0)v_1, v_2)$ satisfies condition 4° of Theorem 1, and condition 4° holds for any R ; there exists a number $K > 0$ such that, for all $t \in [0, T]$, $v_1, v_2 \in H$, the estimate holds

$$\operatorname{Re}(f(t, A^{-1/2}(0)v_1, v_2), v_2) \leq K(1 + \|v_1\|^2 + \|v_2\|^2).$$

Then problem (4) has a unique solution on $[0, T]$.

Theorem 3 for particular cases of equation (4) was proved in (9-11).

Consider the Cauchy problem for the linear equation

$$u''(t) + A(t)u'(t) + B(t)u(t) + C(t)u(t) = f(t); \quad u(0) = u_0, \quad u'(0) = u_1 \quad (6)$$

in a Banach space E .

Theorem 4. Let the following conditions be fulfilled:

1°. The operator $B^{1/2}(t)$ has an everywhere dense domain of definition independent of t , $D(B^{1/2}(t)) = D(B^{1/2})$, and for some $\omega \geq 0$ the estimate

$$\|R(i\lambda, B^{1/2}(t))\| \leq \frac{1}{|\lambda| - \omega}$$

holds for all real λ , $|\lambda| > \omega$.

2°. The operator-function $B^{1/2}(t)B^{-1/2}(0)$ is twice strongly continuously differentiable on $[0, T]$; the operator-functions $A(t)$, $C(t)B^{-1/2}(0)$ are strongly continuously differentiable on $[0, T]$.

3°. $f(t)$ is continuously differentiable on $[0, T]$.

4°. $u_0 \in D(B(0))$, $u_1 \in D(B^{1/2})$.

Then problem (6) has a unique solution on $[0, T]$.

Problem (6) in Hilbert space has been considered by other methods in ^(12–14). We note, however, that Theorem 4 is new also for the case of Hilbert space and does not follow from the results of ^(12–14).

3. As an application, let us consider in the space $L_2([0, T] \times \Omega)$ the first boundary-value problem for a quasilinear system of hyperbolic type

$$\partial^2 U(t, x) / \partial t^2 + L(t, x, D_x)U(t, x) = f(t, x, U, D_{xU}, \dots, D_x^{kU}, D_{tU}) \quad (7)$$

with the conditions

$$U|_{t=0} = U_0(x), \quad U'|_{t=0} = U_1(x), \quad U|_{\Gamma} = \frac{\partial U}{\partial n} \Big|_{\Gamma} = \dots = \frac{\partial^{m-1} U}{\partial n^{m-1}} \Big|_{\Gamma} = 0. \quad (8)$$

Here $U(t, x)$ is an N -dimensional vector-function.

$$L(t, x, D_x)U(x) = (-1)^m \sum_{|i|, |j| \leq m} D^i A^{ij}(t, x) D^j U(x)$$

is a strongly elliptic operator, symmetric in the sense of Lagrange. If the number of spatial variables is $2n-1$ and $n \leq m$, then $k \leq m-n$; if, however, the number of spatial variables is $2n$ and $n \leq m-1$, then $k \leq m-n-1$. Applying the results of ^(15–17), as well as the theorems of this article, one proves local existence and uniqueness of the solution of problem (7), (8), if the function $f(t, x, v_1, \dots, v_s)$ is continuous together with its derivatives with respect to t and all v_i in the domain $\{t \in [0, T], x \in \bar{\Omega}, |v_i| \leq R, i = 1, \dots, s\}$ (R is some positive number), and these derivatives satisfy a Lipschitz condition in all v_i . If the function $f(t, x, v_1, \dots, v_s)$, in addition to the listed smoothness conditions, has the linear majorant

$$|f(t, x, v_1, \dots, v_s)| \leq C \left(1 + \sum_{i=1}^s |v_i| \right),$$

then a nonlocal theorem of existence and uniqueness of the solution of problem (7), (8) is proved. The fact that in the abstract theorems formulated in the

preceding sections the constancy of the domain of definition is required only for $A^{1/2}(t)$ (and not for $A(t)$) makes it possible to investigate also other boundary-value problems for equation (7).

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