

ZERO-DIMENSIONAL STATIONARY SUBSETS OF FINITE-DIMENSIONAL METRIC SPACES

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Abstract

Full Text

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MATHEMATICS

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ZERO-DIMENSIONAL STATIONARY SUBSETS OF FINITE-DIMENSIONAL METRIC SPACES

(Presented by Academician P. S. Aleksandrov on 1 II 1966)

Definition. A subset A of a metrizable topological space (X, T) will be called **stationary** if there exists on the set X a metric ρ compatible with the topology T (i.e., a metric satisfying the condition $T_\rho = T$, where T_ρ is the topology induced by the metric ρ), and moreover the following condition is fulfilled:

(U). For every continuous mapping $f : X \rightarrow X$ for which $\rho(x, f(x)) < 1$ for all points $x \in X$, the relation

$$A \cap f(X) = \emptyset$$

holds.

Let us note that if the space (X, ρ) is a compact metric space, then the set $A \subset X$ is stationary in the topological space (X, T_ρ) if and only if there exists a number $\varepsilon > 0$ such that, for every continuous mapping $f : X \rightarrow X$ satisfying the condition $\rho(x, f(x)) < \varepsilon$ for all points $x \in X$, one has

$$A \cap f(X) \neq \emptyset.$$

In this note we prove the existence of "small" stationary sets in the sense indicated below.

Theorem. *In every finite-dimensional (in the sense of dim) metrizable space (X, T) there exists a zero-dimensional closed stationary subset.*

Proof. Let $\dim X = n$, where n is a natural number (the cases $n = 1$ and $n = 0$ are excluded as trivial). Then, as is known, there exist closed sets $A_i \subset X$, $i = \pm 1, \pm 2, \dots, \pm n$, such that for any system of closed partitions F_i between the sets A_{-i} and A_i , $i = 1, 2, \dots, n$, we have

$$\bigcap_{i=1}^n F_i \neq \emptyset.$$

But, on the other hand, since the space (X, T) is metrizable, $\dim X = \text{Ind } X$, and therefore there exist such closed partitions F_i that

$$\dim \bigcap_{i=1}^k F_i = n - k.$$

In particular,

$$\dim \bigcap_{i=1}^n F_i = 0.$$

We shall prove that the set

$$F = \bigcap_{i=1}^n F_i$$

is stationary.

Indeed, let ρ_0 be any metric on the space X , compatible with the topology T , and let $g_i : X \rightarrow I$ be continuous mappings of the space X into the interval $I = [-1, 1]$ such that $g_i(x) = \varepsilon$ for all points $x \in A_{\varepsilon i}$, $\varepsilon = -1$ or 1 , and $g_i(x) = 0$ for all points $x \in F_i$; finally, $0 < |g_i(x)| < 1$ for all points $x \in X \setminus (A_{-i} \cup F_i \cup A_i)$. Define a new metric on the space X by the equality

$$\rho(x, y) = \rho_0(x, y) + \max_i |g_i(x) - g_i(y)|.$$

The metric ρ is also compatible with the topology T .

Let now $f : X \rightarrow X$ be an arbitrary continuous mapping, with $\rho(x, f(x)) < 1$ for all $x \in X$. Then $g_i f(x) < 0 < g_i f(y)$ for all points $x \in A_{-i}$, $y \in A_i$; hence it follows that the sets $(g_i f)^{-1}(0)$ are partitions between the sets A_{-i} and A_i , $i = 1, 2, \dots, n$. Therefore there exists a point $x \in X$ such that $g_i f(x) = 0$ for all indices $i = 1, 2, \dots, n$. But then $f(x) \in F$, i.e. $F \cap f(x) \neq \emptyset$. The theorem is proved.

This result can be generalized.

Definition*. A subset A of a topological space (X, T) will be called **stationary** if there exists on it a continuous pseudometric ρ for which condition $(\bar{y})$ is satisfied.

Then we have

Theorem*. *In every perfectly normal space (X, T) for which $\dim X = \text{Ind } X < \infty$, there exists a closed stationary zero-dimensional subset.*

Problem. Does there exist a metrizable space that contains no zero-dimensional closed stationary subset? Is the Hilbert brick $I^{\aleph_0}$ such a space?

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Note: Figure translations are in progress. See original paper for figures.

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