

ON THE DENSITY OF FINITE FUNCTIONS IN $\backslash(l_p^{\{1)\}}\backslash$

S. L. Sobolev

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Abstract

Full Text

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MATHEMATICS

Academician S. L. SOBOLEV

ON THE DENSITY OF FINITE FUNCTIONS IN $l_p^{(l)}$

In ^(1,2) a theorem is proved and used on the density of finite functions in spaces of functions defined on nets. The proof given there is based, on the one hand, on the interpolation theorem of Raben' kii-Fillipov ⁽³⁾, and on the other, on the consideration of certain functional spaces $M_p^{(l)}(N)$, introduced by the author for this purpose. For the case when $lp > n$, i.e. for spaces imbedded in C , the question can be solved more simply.

In the present note a new proof of this theorem is given, not requiring the additional consideration of new spaces.

We shall prove the following theorem, which is also of independent interest:

Theorem 1. Let $\varphi(x)$ be a function of the n -dimensional vector $x \in E_n$, belonging to the space $L_p^{(l)}$ for $lp > n$. The values of this function at the points of an arbitrary regular net

$$\varphi[\beta] = \varphi(H\beta), \quad (1)$$

where H is a matrix with determinant equal to unity, and β is an integer vector

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix},$$

form an element of the space $l_p^{(l)}$ with norm

$$\|\varphi[\beta]\|_{l_p^{(l)}} = \left\{ \sum_{\beta} \sum_{|\alpha|=l} |\Delta^{[\alpha]} \varphi[\beta]|^p \right\}^{1/p}. \quad (2)$$

The proof of this theorem is based on the direct solution of a problem of the calculus of variations.

First of all we note that, for our purposes, it is sufficient to restrict ourselves to a cubic net, since by an affine transformation any net can be transformed into

a cubic one, and such a transformation preserves $L_p^{(l)}$, replacing the norm by an equivalent one.

We decompose our space E_n into elementary cubes with centers at the vertices of the net:

$$\Omega_\beta = \{x \in E_n : |x_j - \beta_j| < 1/2\}. \quad (3)$$

Corresponding to each integer vector $\beta^{(0)}$, we construct a domain $S_{\beta^{(0)}}$ from such cubes, belonging to the elementary coordinate simplex

$$S_{\beta^{(0)}} = \bigcup \Omega_\beta, \quad \beta = \beta^{(0)} + \gamma, \quad 0 \leq \gamma, \quad \Sigma \gamma_j \leq l. \quad (4)$$

The number of integer points inside the domain $S_{\beta^{(0)}}$ will be $M = (l+n)!/l!n!$. We renumber these points

$$\beta^{(j)} = \beta^{(0)} + \gamma^{(j)}, \quad j = 1, 2, \dots, M. \quad (5)$$

With the aid of these points let us compute all finite differences up to order m inclusive at the point $\beta^{(0)}$ of a certain function $\psi(x)$

$$\Delta^{[\alpha]} \psi(x) = \Delta_1^{[\alpha_1]} * \Delta_2^{[\alpha_2]} * \dots * \Delta_n^{[\alpha_n]} * \psi(x),$$

$$\Delta_j^{[\alpha_j]} = \underbrace{\Delta_j * \Delta_j * \dots * \Delta_j}_{\alpha_j}, \quad (6)$$

where $\Delta_j * \psi = \psi(x + i_j) - \psi(x)$, and i_j is the unit vector directed along the x_j -axis. The number of such differences will also be equal to M , and we can number them by means of some enumeration of the vectors $\alpha^{(j)}$, where

$$\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \dots, \alpha_n^{(j)}), \quad \alpha_s^{(j)} \geq 0, \quad |\alpha^{(j)}| = \sum_{k=1}^n \alpha_k^{(j)} \leq l. \quad (7)$$

Let us assume that this enumeration is such that the differences of exactly order l have numbers from 1 to K inclusive, where $K = (l+n-1)!/(n-1)l!$. Introduce the space E_M , whose elements are the vectors $\eta = (\eta_1, \eta_2, \dots, \eta_M)$. For each $S_{\beta^{(0)}}$ we shall solve the following variational problem.

Find

$$\min \|\psi\|_{L_p^{(l)}(S_{\beta^{(0)}})} = \rho(\eta) \quad (8)$$

under the conditions

$$\Delta^{[\alpha^j]} \psi_{\beta(0)} = \eta_j. \quad (9)$$

It is solved in the same way as in the author's book ⁽⁴⁾.

With the aid of direct methods of the calculus of variations it is established that the desired minimum is attained at some function $\psi_{\beta(0)}(x) \in L_p^{(l)}$, which will be continuous by virtue of the embedding theorems. For this purpose one constructs a minimizing sequence which, by the uniform convexity of $L_p^{(l)}$, proved already by Clarkson, turns out to converge to a function $\psi_{\beta(0)}(x | \eta)$, giving the norm of ψ a minimum under conditions (9). The limiting function $\rho(\eta)$ will be continuous, as is easily verified by giving $\psi_{\beta(0)}(x | \eta)$ a variation $\zeta = \sum \Delta \eta_j \xi_j(x)$, small in norm and such that $\psi_{\beta(0)}(x | \eta) + \zeta$ satisfies conditions (9) with right-hand side $(\eta + \Delta \eta)$. Thus it is proved that $\rho(\eta + \Delta \eta) - \rho(\eta) < \varepsilon$ for sufficiently small $\Delta \eta$.

One can also establish that the function $\rho(\eta)$ depends only on the differences of exactly order l and does not depend on differences of lower order, i.e., is a function of the variable $\eta(\eta_1, \eta_2, \dots, \eta_K)$ in E_K . This follows from the fact that adding to ψ a polynomial of degree $l-1$ does not change the norm.

The minimum of the function $\rho(\eta)$ on the surface σ

$$\sigma = \left\{ \eta \in E_k : \sum_{|\alpha|=m} |\eta_\alpha|^p = 1 \right\}$$

is a positive constant,

$$\min \rho(\eta) = \Lambda > 0.$$

It is easy to establish the inequality

$$\|\varphi\|_{L_p^{(l)}(S_{\beta(0)})}^p \geq \Lambda^p \sum_{|\alpha|=l} |\Delta^{[\alpha]} \varphi|^p.$$

From this inequality, summing over all $\beta(0)$ and noting that

$$\sum_{\beta(0)} \|\varphi\|_{L_p^{(l)}(S_{\beta(0)})}^p = M \|\varphi\|_{L_p^{(l)}(E_n)}^p,$$

we shall have

$$M \|\varphi\|_{L_p^{(l)}(S_{\beta(0)})}^p \geq \Lambda^p \|\varphi\|_{L_p^{(l)}}^p,$$

which proves our theorem.

Using this theorem and the Rjabenkij-Filippov interpolation process, one can obtain a proof of the theorem on the density of finitely supported functions in $l_p^{(l)}$ by the same method as in paper ⁽¹⁾, replacing at first $\varphi[\beta]$ by an interpolation function and then taking a sequence of finitely supported functions approximating it.

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Note: Figure translations are in progress. See original paper for figures.

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