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Abstract

Full Text

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MATHEMATICS

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ON SOME CONDITIONAL EXTREMAL PROBLEMS IN THE CLASS OF ENTIRE FUNCTIONS OF FINITE DEGREE

(Presented by Academician S. N. Bernstein on 7 V 1965)

Denote by $\dot{W}_\sigma^{(p)}$ ($p \geq 1$) the class of real entire functions $\varphi_\sigma(z)$ of degree σ , nonnegative on the real axis, with norm

$$\|\varphi_\sigma\|_{L_p} = \left(\int_{-\infty}^{\infty} [\varphi_\sigma(x)]^p dx \right)^{1/p} < \infty. \quad (1)$$

Let $F_j(\varphi_\sigma)$ ($j = 1, \dots, \nu$) be linear functionals defined on the set $\dot{W}_\sigma^{(p)}$.

We shall consider functions $\varphi_\sigma(x)$ subject to the conditions

$$F_j(\varphi_\sigma) = A_j \quad (j = 1, \dots, \nu), \quad (2)$$

where A_j are given real numbers, with not all A_j equal to zero; the conditions (2) are assumed admissible, i.e. compatible and not contradicting the membership of the functions $\varphi_\sigma(x)$ in the class $\dot{W}_\sigma^{(p)}$ ($p \geq 1$).

In the present article upper and lower estimates are indicated for the quantity

$$\|\varphi_\sigma^*\|_{L_p} = \inf_{\varphi_\sigma \in \dot{W}_\sigma^{(p)}} \|\varphi_\sigma\|_{L_p}. \quad (3)$$

The function $\varphi_\sigma^*(x) \in \dot{W}_\sigma^{(p)}$ will be called the extremal function of our problem in the class $\dot{W}_\sigma^{(p)}$. The founder of this type of problem in the metric $C(-\infty, \infty)$ is S. N. Bernstein ⁽¹⁾.

It is obvious that $\varphi_\sigma^* \neq \text{const}$, since otherwise one of the conditions (1) or (2) would not be satisfied.

It is known ^(1,2) that if $\varphi_\sigma(x) \in \dot{W}_\sigma^{+(p)}$ and $p_1 > p \geq 1$, then an inequality of S. M. Nikolskii type holds:

$$\|\varphi_\sigma\|_{L_{p_1}} \leq \left(\frac{s\sigma}{2\pi}\right)^{1/p-1/p_1} \|\varphi_\sigma\|_{L_p}, \quad (4)$$

where $s = \lceil -p/2 \rceil$ is the least integer not less than $p/2$. Therefore, denoting by $\Phi_\sigma(x)$ a function belonging to $\dot{W}_\sigma^{+(1)}$, and taking into account that $\dot{W}_\sigma^{+(1)} \subset \dot{W}_\sigma^{+(p)} \subset \dot{W}_\sigma^{+(\infty)}$, we have

$$\|\varphi_\sigma^*\|_{L_p} \leq \|\Phi_\sigma^*\|_{L_p} \leq (\|\Phi_\sigma^*\|_C)^{(p-1)/p} (\|\Phi_\sigma^*\|_L)^{1/p}, \quad (5)$$

where a function $f_\sigma(z) \in \dot{W}_\sigma^{+(\infty)}$ is considered bounded on $(-\infty, \infty)$.

Consequently, the following is valid.

Theorem. If $f_\sigma^*(x)$ and $\Phi_\sigma^*(x)$ are the extremal functions of our problem, respectively, in the classes $\dot{W}_\sigma^{+(\infty)}$ and $\dot{W}_\sigma^{+(1)}$, then in the class $\dot{W}_\sigma^{+(p)}$ ($1 < p < \infty$) there exists a function $\varphi_\sigma^*(x)$, subject to the conditions (2),

with the least norm $\|\varphi_\sigma^*\|_{L_p}$, for which the inequalities

$$\left(\frac{2\pi}{s\sigma}\right)^{1/p} |f_\sigma^*|_C \leq \|\varphi_\sigma^*\|_{L_p} \leq \|\Phi_\sigma^*\|_{L_p} \leq (\|\Phi_\sigma^*\|_C)^{(p-1)/p} (\|\Phi_\sigma^*\|_L)^{1/p}. \quad (6)$$

As is known ⁽⁴⁾, if $\Phi_\sigma(x) \in \dot{W}_\sigma^{+(1)}$, then for $\nu \leq 2$

$$\Phi_\sigma^*(x) = [\Psi_{\sigma/2}(x)]^2, \quad (7)$$

where $\Psi_{\sigma/2}(x)$ is a certain real entire function of degree $\sigma/2$ from the class $W_\sigma^{(2)}$ (see ⁽²⁾, p. 38). Then, by the well-known Wiener-Paley theorem, we have

$$\Psi_{\sigma/2}(x) = \int_{-\sigma/2}^{\sigma/2} \gamma(t) e^{ixt} dt, \quad (8)$$

where $\gamma(t) \in L_2[-\sigma/2, \sigma/2]$, and

$$\int_{-\infty}^{\infty} [\Psi_{\sigma/2}(x)]^2 dx = 2\pi \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2 dt. \quad (9)$$

Writing the formal expansion of the function $\gamma(t)$ in the Legendre polynomials $\{\hat{P}_n(t)\}_0^\infty$ normalized on $[-\sigma/2, \sigma/2]$,

$$\gamma(t) \sim \sum_{k=0}^{\infty} b_k \hat{P}_k(t), \quad (10)$$

where

$$b_k = \int_{-\sigma/2}^{\sigma/2} \gamma(t) \hat{P}_k(t) dt \quad (k = 0, 1, 2, \dots), \quad (11)$$

we obtain

$$\|\Phi_\sigma^*\|_L = \int_{-\infty}^{\infty} [\Psi_{\sigma/2}(x)]^2 dx = 2\pi \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2 dt = 2\pi \sum_{k=0}^{\infty} b_k^2. \quad (12)$$

Moreover,

$$\|\Phi_\sigma^*\|_C = \left[\sup_{-\infty < x < \infty} |\Psi_{\sigma/2}(x)| \right]^2 \leq \left(\int_{-\sigma/2}^{\sigma/2} |\gamma(t)| dt \right)^2 \leq \sigma \int_{-\sigma/2}^{\sigma/2} |\gamma(t)|^2 dt = \sigma \sum_{k=0}^{\infty} b_k^2. \quad (13)$$

Finally, denote by $\widetilde{B}_\sigma$ the class of entire functions $g_\sigma(z)$ belonging to the class B_σ (B_σ is the class of entire functions of degree σ , bounded on the entire real axis ⁽¹⁾, and satisfying the conditions (2)). Let $g_\sigma^*(x)$ be an entire function from the class $\widetilde{B}_\sigma$ with the least norm,

$$\|g_\sigma^*\|_C = \inf_{g_\sigma \in \widetilde{B}_\sigma} \|g_\sigma\|_C.$$

It is not difficult to observe that $\|f_\sigma^*\|_C \geq |f_\sigma^*(0)|$ and, moreover,

$$|g_\sigma^*(0)| \leq \|g_\sigma^*\|_C \leq \|f_\sigma^*\|_C. \quad (14)$$

Thus, inequality (6) (taking into account (12), (13), and (14), for $\nu \leq 2$) takes the form

$$\left(\frac{2\pi}{s\sigma} \right)^{1/p} \|g_\sigma^*\|_C \leq \|\varphi_\sigma^*\|_{L_p} \leq \sigma \left(\frac{2\pi}{\sigma} \right)^{1/p} \sum_{k=0}^{\infty} b_k^2. \quad (15)$$

Let us now consider some particular cases of inequalities (6) or (15).

1. Let $\nu = 1$ and let relation (2) be

$$\varphi_\sigma(0) = 1. \quad (16)$$

S. N. Bernstein¹ proved that in this case $g_\sigma^*(x) = \sin \sigma x / \sigma x$ is an extremal function in the class $\widetilde{B}_\sigma$ and $\|g^*\|_C = 1$. On the other hand, as shown in², in this case

$$b_0 = \frac{1}{\sqrt{\sigma}}, \quad b_k = 0 \quad (k = 1, 2, \dots); \quad \Phi_\sigma^*(x) = \left(\frac{\sin \sigma x / 2}{\sigma x / 2} \right)^2. \quad (17)$$

Therefore inequality (15) takes the form

$$(2\pi/s\sigma)^{1/p} \leq \|\varphi_\sigma^*\|_{L_p} \leq (2\pi/\sigma)^{1/p}. \quad (18)$$

2. Let now $\nu = 2$, and let the relations (2) have the form

$$\varphi_\sigma(0) = 1, \quad \varphi'_\sigma(0) = 0. \quad (19)$$

Then for $\Phi_\sigma^*(x) \in W_\sigma^{+(1)}$ (under conditions (19)) the coefficients b_k are determined by the same formulas (17).

On the other hand, it is known³ that in the class $\widetilde{B}_\sigma$, under conditions (19), the extremal function is $\cos \sigma x$, with norm equal to 1. Therefore, as in the preceding case, we have

$$(2\pi/s\sigma)^{1/p} \leq \|\varphi_\sigma^*\|_{L_p} \leq (2\pi/\sigma)^{1/p}. \quad (18')$$

It is interesting to note that for $p = 2$ (then $s = 1$) inequalities (18) and (18') become the equality

$$\|\varphi_\sigma^*\|_{L_2} = (2\pi/\sigma)^{1/2};$$

moreover, for all $p \in [1, 2]$ the equality

$$\|\varphi_\sigma^*\|_{L_p} = (2\pi/\sigma)^{1/p}$$

holds.

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¹S. N. Bernstein, *Collected Works*, 1, 1952, p. 22.

²B. A. Rymarenko, DAN, 161, No. 4 (1965).

³N. I. Akhiezer, *Matematicheskii sbornik*, 31 (73), No. 2 (1952).

CITED LITERATURE

Note: Figure translations are in progress. See original paper for figures.

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