

ON THE MUTUAL GROWTH OF THE COEFFICIENTS OF ONE CLASS OF $\backslash(p\backslash)$ -VALENT FUNCTIONS

E. G. Goluzina

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Abstract

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MATHEMATICS

E. G. GOLUZINA

ON THE MUTUAL GROWTH OF THE COEFFICIENTS OF ONE CLASS OF p -VALENT FUNCTIONS

(Presented by Academician V. I. Smirnov, 19 XI 1965)

For the class $S^\square$ of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular and univalent in the mean with respect to area in the disk $|z| < 1$, Hayman ⁽¹⁾ proved that

$$||a_{n+1}| - |a_n|| < A, \quad n \geq 2, \quad (1)$$

where A is an absolute constant. The order of estimate (1) is sharp.

For the class S^* of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, univalent and starlike in the disk $|z| < 1$, estimate (1) (with $A < 100$) was obtained by G. M. Goluzin ⁽²⁾ in 1946.

Hayman ⁽³⁾ posed the question whether

$$||a_{n+1}| - |a_n|| \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

for every function of the class $S^\square$, other than functions of a certain special form.

In this direction, for the class S^* the following result of Pommerenke ⁽⁴⁾ is known: every function $f(z) \in S^*$ either has the form

$$\frac{z}{(1 - e^{-i\theta_1} z)(1 - e^{-i\theta_2} z)}, \quad \theta_1, \theta_2$$

real, or there exists $\delta = \delta(f) > 0$ such that

$$|a_{n+1}| - |a_n| = O(n^{-\delta}).$$

Lucas ⁽³⁾ generalized estimate (1) to the class of functions of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, regular and p -valent in the mean with respect to area ($p \geq 1$) in the disk $|z| < 1$, proving that

$$||a_{n+1}| - |a_n|| < A(p)\mu_p n^{2p-2}, \quad n \geq 1, \quad (2)$$

where $\mu_p = \max_{0 \leq \nu \leq p} |a_\nu|$, $A(p)$ is a constant depending only on p . The order of estimate (2) is sharp.

In the present paper, for a certain class of p -valent functions, a result analogous to the above-mentioned result of Pommerenke is obtained.

Let $F(p)$ (p a fixed natural number) be the class of functions $g(z)$ representable in the disk $|z| < 1$ by the formula

$$g(z) = [\varphi(z)]^p z^{q-p} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right) (1 - z\bar{\alpha}_s),$$

where q is an integer, $1 \leq q \leq p$, $\varphi(z) \in S^*$, $0 < |\alpha_s| < 1$, $s = 1, 2, \dots, p - q$; $F(1) \equiv S^*$.

Let $F(p, q)$ be the subclass of all functions from $F(p)$ of the form

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n$$

with fixed q .

Bender⁽⁵⁾ showed that all functions of the class $F(p, q)$ are p -valent. For $p > q \geq 1$, the class $F(p, q)$ contains as a subclass the class $S(p, q)$ ⁽⁵⁾ of functions of the form

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n,$$

regular in the disk $|z| < 1$ and satisfying the condition: for each function $g(z)$ there exists ρ , $0 < \rho < 1$, such that

$$\operatorname{Re} \left[\frac{zg'(z)}{g(z)} \right] > 0, \quad \rho < |z| < 1,$$

$$\int_0^{2\pi} \operatorname{Re} \left[\frac{zg'(z)}{g(z)} \right] d\theta = 2\pi p, \quad z = re^{i\theta}, \quad \rho < r < 1.$$

For $p = q$ we have $F(p, p) \equiv S(p, p)$.

Theorem. If

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in F(p, q)$$

and $p > 1$, and if

$$g(z) \not\equiv \frac{z^q}{(1 - e^{-i\theta} z)^{2p}} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right) (1 - z\bar{\alpha}_s),$$

then there exists $\delta = \delta(g) > 0$ such that

$$|a_{n+1}| - |a_n| = O(n^{2p-2-\delta}), \quad n \geq q.$$

It is not difficult to show that there exist functions of the class $F(p, q)$, of the form excluded in the theorem, for which

$$|a_{n+1}| - |a_n| \sim K n^{2p-2} \quad \text{as } n \rightarrow \infty,$$

where K is a constant independent of n .

In the proof of the theorem the following result is established:

Lemma. Let $m \geq 1$, m an integer, and let

$$g(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \in F(p, q).$$

If

$$g(z) \not\equiv z^q \prod_{k=1}^{m+1} (1 - e^{-i\theta_k} z)^{-2p/(m+1)} \prod_{s=1}^{p-q} \left(1 - \frac{z}{\alpha_s}\right) (1 - z\bar{\alpha}_s)$$

(θ_k real), then there exist complex numbers c_k , $k = 0, 1, \dots, m$, $|c_0| = |c_m| = 1$, and $\delta > 0$, depending only on m and g , such that for $n \geq q$ we have

$$|c_0 n^2 a_n + c_1 (n+1)^2 a_{n+1} + \dots + c_m (n+m)^2 a_{n+m}| = O(n^{2/(m+1)+2p-1-\delta}).$$

For $p = 1$, an analogous result was obtained by Pommerenke ⁽⁴⁾ for the class of close-to-convex functions.

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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