

DISTRIBUTION OF EIGENVALUES FOR SINGULAR DIFFERENTIAL OPERATORS

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Abstract

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MATHEMATICS

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DISTRIBUTION OF EIGENVALUES FOR SINGULAR DIFFERENTIAL OPERATORS

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I. Consider a strongly elliptic operator L in the space of N -dimensional vector-functions $u(x) = \{u_1(x), \dots, u_N(x)\}$

$$L = (-1)^m \sum_{j_1 + \dots + j_n = 2m} A^{j_1 \dots j_n}(x) \frac{\partial^{2m}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} + L_1 \left(x, \frac{\partial}{\partial x} \right) + Q(x) =$$

$$= L_0 + L_1 + Q(x),$$

where $x = (x_1, \dots, x_n)$, $-\infty < x_j < \infty$, $j = 1, \dots, n$, $A^{j_1 \dots j_n}(x)$ are symmetric matrices. By $L_1(x, \partial/\partial x)$ is denoted a linear differential operator of order $< 2m$, generally speaking with complex-valued coefficients; $Q(x)$ is the operator of multiplication by the symmetric matrix $Q(x)$.

Number the characteristic roots $\lambda_i(x)$ of the matrix $Q(x)$ in increasing order:

$$\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_n(x).$$

We shall assume that the following conditions are satisfied:

a) the matrix $A(s, x)$, defined by the equality

$$A(s, x) = (-1)^{m+1} \sum_{j_1 + \dots + j_n = 2m} A^{j_1 \dots j_n}(x) (is_1)^{j_1} \dots (is_n)^{j_n},$$

is Hermitian-symmetric for all x and real $s = (s_1, \dots, s_n)$; its characteristic roots $\sigma_j(s, x)$ satisfy the inequality

$$\sigma_j(s, x) \leq -\delta |s|^{2m}, \quad \delta > 0;$$

b) $\|A(\xi, x) - A(\xi, y)\| \leq c|x - y|^\gamma |\xi|^{2m}$, $0 < \gamma < 1$, for $|x - y| \leq 1$.

The characteristic roots $\lambda_i(x)$ of the matrix $Q(x)$ are subject to the conditions:

- 1) $\|Q(x) - Q(y)\| \leq B\lambda_1^\alpha(y)|x - y|$, if $|x - y| \leq 1$, $\alpha < 1 + 1/2m$;
- 2) $\lambda_n(x) \leq B_1 \exp(c^*|x - y|^{1/2m}(y))$, if $|x - y| > 1$; here c^* is some constant depending on the matrix $A(\xi, x)$;
- 3) $\lambda_1(x) \leq B\lambda_1(y)$, if $|x - y| \leq 1$;
- 4) $\lambda_n(x) \leq \lambda_1^k(x)$ for some $k \leq 1$;
- 5) the elements $p_{ik}^{j_1 \dots j_n}(x)$ of the matrix $P^{j_1 \dots j_n}(x)$ ($j_1 + \dots + j_n = |j|$) of the operator $L_1(x, \partial/\partial x)$, standing at $\partial^{|j|}/\partial x_1^{j_1} \dots \partial x_n^{j_n}$, satisfy the condition

$$|p_{ik}^{j_1 \dots j_n}(x)| \leq \lambda_1^{(2m-|j|)/2m-\varepsilon_0}(x), \quad \varepsilon_0 > 0.$$

We shall be interested in asymptotic formulas for the function $N(\lambda)$ —the number of eigenvalues of the operator L not exceeding λ . With this

for this purpose one can study the Green function (more precisely, the Green matrix-function) $G(x, y, t)$ of the Cauchy problem for the corresponding parabolic system:

$$\partial u / \partial t = -Lu = -(L_0 + L_1 + Q(x))u. \quad (1)$$

The function $N(\lambda)$ will be connected, as is easily verified, with $G(x, y, t)$ in the following way:

$$\text{tr} \int_{-\infty}^{\infty} G(x, x, t) dx = \int_0^{\infty} e^{-\lambda t} dN(\lambda).$$

Thus it is necessary to study $G(x, y, t)$ for small t , uniformly in R_n . For a single elliptic equation this question was studied by the author in ⁽¹⁾.

Let $G_0(x-y, \eta, t)$ denote the Green function of the following system with “frozen” coefficients:

$$\frac{\partial U}{\partial t} = (-1)^{m+1} \sum_{|j|=2m} A^{j_1 \dots j_n}(\eta) \frac{\partial^{2m} U}{\partial x_1^{j_1} \dots \partial x_n^{j_n}} - Q(\eta)U.$$

Theorem 1. *If conditions a), b) and 1)–5) are satisfied, then for the Green function $G(x, y, t)$ of system (1) one has the representation*

$$G(x, y, t) = G_0(x - y, y, t) + S(x, y, t), \quad (2)$$

where the matrix $S(x, y, t)$ satisfies the inequality

$$\|S(x, y, t)\| \leq \frac{c_1}{t^{(n-\varepsilon)/2m}} \exp \left(-ct\lambda_1(y) - c \frac{|x-y|^{2m'}}{t^{1/(2m-1)}} \right) + \frac{c_1}{\lambda_1^l(y)} \exp \left(-c \frac{|x-y|^{2m'}}{t^{1/(2m-1)}} \right).$$

Here $1/2m + 1/2m' = 1$; the number $l > 0$ can be taken arbitrarily large; c, c_1 are constants.

The proof of the theorem is carried out according to the scheme of the author's work ⁽¹⁾.

We note that, although representation (2) contains rather rich information, for all the systems under consideration one cannot obtain more or less explicit formulas for the principal term of $N(\lambda)$. This is connected with the fact that the matrices $A(s, x)$ and $Q(x)$ do not commute with one another. We shall now consider two special cases.

Theorem 2. Let the self-adjoint operator L have the form

$$Ly = (-1)^m y^{(2m)} + L_1 \left(x_1, \frac{d}{dx} \right) y + Q(x)y,$$

where $y(x) = \{y_1(x), \dots, y_n(x)\}$, $x \in R_1$, the coefficients of the operator L_1 and the matrix $Q(x)$ satisfy conditions 1)–5). In addition, $\lambda^{-r}(x) \in L_1(-\infty, \infty)$ for some $r > 0$, and the function $M(\lambda)$, defined by the equality

$$M(\lambda) = \frac{1}{\pi} \sum_{i=1}^n \int_{\lambda_i(x) < \lambda} (\lambda - \lambda_i(x))^{1/2m} dx$$

satisfies some Tauberian condition *. Then the formula holds:

$$N(\lambda) \sim \frac{1}{\pi} \sum_{i=1}^n \int_{\lambda_i(x) < \lambda} (\lambda - \lambda_i(x))^{1/2m} dx.$$

* For example, the Tauberian condition in the theorem of B. M. Korenblum ⁽²⁾ $\lambda M'(\lambda) < a_0 M(\lambda)$.

The proof of this theorem follows from Theorem 1, since in this case

$$\text{tr} \int_{-\infty}^{\infty} G_0(0, x, t) dx = \frac{1}{2\pi} \frac{1}{t^{1/2m}} \left\{ \int_{-\infty}^{\infty} e^{-s^{2m}} ds \right\} \text{tr} \int_{-\infty}^{\infty} e^{-Q(x)t} dx = \int_{-\infty}^{\infty} e^{-\lambda t} dM(\lambda).$$

Theorem 3. Let the self-adjoint operator L have the form

$$Ly = (-1)^m A \frac{d^{2m}y}{dx^{2m}} + L_1 \left(x, \frac{d}{dx} \right) y + x^{sRy},$$

where $x \in R_1$; A, R are positive definite constant matrices. The coefficients $p^j(x)$ of the operator $L_1(x, d/dx)$, which has order $< 2m$, standing at the j -th derivative, satisfy the inequality

$$\|p^j(x)\| \leq c_j |x|^{s(2m-|j|)/2m-\varepsilon_0}, \quad \varepsilon_0 > 0,$$

for large x . Then the formula holds:

$$N(\lambda) = \frac{S}{2\pi\Gamma(1/2m + 1/s + 1)} \lambda^{1/2m+1/s}, \quad S = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} \text{tr} \exp\{-\xi^{2m} A - \sigma^s R\} d\xi.$$

Condition 1°, which the matrix $Q(x)$ must satisfy, is very restrictive, since it essentially shows that the largest root can grow only in a definite way, in accordance with the growth of $\lambda_1(x)$. Roughly speaking, if $\lambda_n(x) \leq c\lambda_1^k(x)$, then $k < 1 + 1/2m$.

Theorem 4. Let the matrix $Q(x)$ of the operator L (self-adjoint)

$$L = (-1)^m \frac{d^{2m}}{dx^{2m}} + L_1 \left(x, \frac{d}{dx} \right) + Q(x)$$

be such that the elements of the unitary matrix $T(x)$ reducing Q to diagonal form have $2m$ bounded derivatives for all $x \in R_1$. Then the formula holds

$$N(\lambda) \sim \frac{1}{\pi} \sum_{i=1}^n \int_{\lambda_i(x) < \lambda} \{\lambda - \lambda_i(x)\}^{1/2m} dx,$$

if the following conditions are fulfilled:

- 1) $|\lambda_i(x) - \lambda_i(y)| \leq B\lambda_i^\alpha(y)|x - y|$, $\lambda_i(x) \leq c\lambda_i(y)$ for $|x - y| \leq 1$, $\alpha < 1 + 1/2m$;
- 2) $\lambda_i(x) \leq B \exp(c^* \lambda_i^{1/2m}(y)|x - y|)$ for $|x - y| > 1$ and some c^* ;
- 3) $\lambda_i^{-l}(x) \in L_1(-\infty, \infty)$ for some $l > 0$;
- 4) the Tauberian condition is satisfied for the function $M(\lambda)$;
- 5) for the coefficients $p^j(x)$ of the operator $L_1(x, d/dx)$ the subordination condition is satisfied:

$$\|p^j(x)\| \leq c\lambda_1(x)^{(2m-|j|)/2m-\varepsilon_0}, \quad \varepsilon_0 > 0.$$

II. In [1] and above, the case was considered where only the “free” coefficient affected the asymptotics of $N(\lambda)$. The coefficients standing at the other lower derivatives played a subordinate role. We now formulate one result pertaining to another situation.

Consider the self-adjoint ordinary operator L

$$Ly = (-1)^m y^{(2m)} + p_{2m-2}(x)y^{(2m-2)} + \dots + p_0(x)y + L_1 \left(x, \frac{d}{dx} \right) y,$$

where $y(x)$ is a scalar function, $x \in R_1$.

Assume that the characteristic polynomial

$$P(s, x, \lambda) = s^{2m} + p_{2m-2}(x)(is)^{2m-2} + \dots + p_0(x) + \lambda$$

for all sufficiently large x and $\lambda > 0$ and all s are positive. Further suppose that the coefficients $p_i(x)$ of the operator L and the roots $s_i(x, \lambda)$ of the characteristic polynomial $P(s, x, \lambda)$ are such that the following conditions are satisfied:

$$1^\circ. \quad B \leq \left| \frac{s_k(x, \lambda)}{s_j(x, \lambda)} \right| \leq A, \quad B \leq \left| \frac{\operatorname{Im} s_k(x, \lambda)}{s_j(x, \lambda)} \right| \leq A.$$

$$2^\circ. \quad |p_j(x) - p_j(\xi)| \leq c|x - \xi| p_j^{\alpha_j}(x), \quad p_j(x) \leq c p_j(\xi) \quad \text{for } |x - \xi| \leq 1, \quad \alpha_j < 1 + \frac{1}{2m - j}.$$

$$3^\circ. \quad |p_j(x)| \leq c \exp \left(c^* |x - \xi| p_0^{1/2m}(\xi) \right) \quad \text{for } |x - \xi| \geq 1 \text{ and some } c^*.$$

$$4^\circ. \quad q_0^{-(2m-1)/2m}(x) \in L_1(-\infty, \infty).$$

$$5^\circ. \quad |P'(s_k(x, \lambda))|^{-1} \leq c(\lambda + p_0(x))^{-(2m-1)/2m}.$$

We now introduce the function

$$\rho(\lambda) = i \sum_{k=1}^m \int_{-\infty}^{\infty} \frac{dx}{P'(s_k(x, \lambda))}.$$

As is seen, $\rho(\lambda)$ is an analytic function in the plane $\lambda = \xi + i\tau$ with a slit along the positive semiaxis. Let $\sigma(t)$ be defined by the equality

$$\sigma(t) = \frac{1}{\pi} \lim_{\tau \rightarrow 0} \int_0^t \operatorname{Im} \{ \rho(-\xi + i\tau) \} d\xi, \quad \xi > 0, \quad \tau > 0.$$

Theorem 5. *Let conditions 1°–5° be satisfied, let the function $\sigma(t)$ satisfy the Tauberian condition, and let the coefficients $q_i(x)$ of the operator $L_1(x, d/dx)$ be subordinate to $p_0(x)$, i.e.*

$$|q_i(x)| \leq c p_0^{(2m-i)/2m-\varepsilon_0}(x), \quad \varepsilon_0 > 0.$$

Then the asymptotic formula holds

$$N(\lambda) \sim \sigma(\lambda).$$

To prove the theorem we find the asymptotics of the kernel $K(x, \xi, \lambda)$ of the resolvent of the operator L for large λ . Then we use a Tauberian theorem.

For the operator of the 4th order

$$Ly = y^{\text{IV}} - 2(p_2(x)y')' + p_0(x)y + L_1\left(x, \frac{d}{dx}\right)y,$$

as is not hard to verify, all conditions 1°, 2°, 7° will be satisfied if, for sufficiently large $|p_2'(x)| \leq c(p_0(x) + \lambda_0)$ for some λ_0 and $c < 1$. In this case

$$N(\lambda) \sim \frac{1}{\pi} \int_{p_0(x) < \lambda} \sqrt{-p_2(x) + \sqrt{p_2^2(x) + (\lambda - p_0(x))}} dx.$$

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CITED LITERATURE

- ¹ A. G. Kostyuchenko, DAN, **158**, No. 1, 41 (1964).
- ² B. I. Korenblyum, DAN, **88**, No. 5, 745 (1958).

Note: Figure translations are in progress. See original paper for figures.

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