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Abstract

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MATHEMATICS

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ON THE QUESTION OF DESCRIPTION OF THE DUALITY FUNCTOR

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The concept of duality of functors, first introduced by D. B. Fuks ⁽¹⁾ for functors in the category of topological spaces with a distinguished point, was subsequently extended ⁽²⁾ to functors in a broad class of categories (in the so-called *D*-categories). In the present article, dual functors are described in a certain sense in a number of categories.

1. Let $\mathcal{T}$ be a category consisting of topological spaces with a distinguished point. We shall consider functors acting from the category $\mathcal{T}$ to the category $\mathcal{G}$ of all topological spaces with a distinguished point (by morphisms in the categories $\mathcal{T}$ and $\mathcal{G}$ we mean continuous mappings taking the distinguished point to the distinguished point). We introduce notation: the set $\text{Hom}(X, Y)$ of all morphisms of a space $X \in \mathcal{G}$ into a space $Y \in \mathcal{G}$ with the compact-open topology will be denoted by $H(X, Y)$, the same set with the topology of pointwise convergence by $H_s(X, Y)$; the set of all (not necessarily continuous) mappings of X into Y taking the distinguished point to the distinguished point by $\text{Map}(X, Y)$. A functor F will be called continuous if the mapping defined by it $F : \text{Hom}(X, Y) \rightarrow \text{Hom}(F(X), F(Y))$ is a continuous mapping $H(X, Y)$ into $H_s(F(X), F(Y))$. We shall assume that all functors under consideration satisfy the condition $F(0_{X,Y}) = 0_{F(X), F(Y)}$ (by $0_{X,Y}$ we denote the mapping of X into Y taking all of X to the distinguished point of the space Y). Define the functor $\Sigma_A(X) = A \otimes X$ as the direct product $A \times X$, factorized by the "coordinate cross" $A \times 0 \cup 0 \times X$; here we assume that a fundamental system of the zero point of $A \otimes X$ obtained from any pair of nonzero points $(a, x) \in A \times X$ consists of sets of the form $U \otimes V$, where U is a neighborhood of the point $a \in A$, V a neighborhood of the point $x \in X$, while a fundamental system of the zero point of $A \otimes X$, obtained from any pair $(a, 0) \in A \times X$ or $(0, x) \in A \times X$, consists of sets of the form $U \otimes X \cup A \otimes V$, where U is a neighborhood of the point O in A , V a neighborhood of the point O in X (this definition of the topology in $\Sigma_A(X)$ differs from that adopted by D. B. Fuks ⁽³⁾, but in the case when A and X are bicomact, the two definitions coincide). To each functor F acting from $\mathcal{T}$ to $\mathcal{G}$, put in correspondence the dual

functor DF , acting from $\mathcal{G}$ to the category of sets with a distinguished point; namely, for any $A \in \mathcal{G}$ define the set $DF(A)$ as the set $\{F \rightarrow \Sigma_A\}$ of mappings of the functor F into the functor Σ_A .

By E and P we shall denote, respectively, the line and the plane (the distinguished point is the origin). We shall assume that the category $\mathcal{T}$ considered by us consists of completely regular spaces and contains the spaces E and P .

Define the set $D_R F(A)$ as the set of mappings of the functor F into the functor Σ_A , if these functors are considered only on the category $\mathcal{R} \subset \mathcal{T}$, consisting of the two spaces E, P and the morphisms $P \rightarrow E$.

Theorem 1. Let F be a continuous functor acting from the category $\mathcal{T}$ to the category $\mathcal{G}$. Then the natural mapping of the set

$DF'(A)$ into the set $D_R F'(A)$ is a one-to-one mapping onto $D_R F'(A)$.

Proof. Denote by $\overline{X}$ the set of morphisms of the space X into the line E , and by A' the space $A \otimes E$. Define the mapping

$$i_X : A \otimes X \rightarrow \text{Map}(\overline{X}, A'),$$

by assigning to each point $a \otimes x \in A \otimes X$ the mapping $\varphi : \overline{X} \rightarrow A'$ that takes a function $\psi \in \overline{X}$ to the point $a \otimes \psi(x) \in A \otimes E = A'$. From the complete regularity of the space X it follows that the mapping i_X is injective. For each morphism $\mu : F(E) \rightarrow A$ we construct a mapping

$$\tilde{\mu}_X : F(X) \rightarrow \text{Map}(\overline{X}, A'),$$

by setting $\tilde{\mu}_X(z) = \psi$, where $\psi(\varphi) = \mu F(\varphi)z$ (here $z \in F(X)$, $\varphi \in \overline{X}$). It is not hard to verify that for any $\lambda \in DF'(A)$ we have

$$i_X \lambda_X = (\widetilde{\lambda_E})_X;$$

using the injectivity of the mapping i_X , we see that the mapping of functors $\lambda \in DF'(A)$ is uniquely determined by the morphism $\lambda_E : F(E) \rightarrow A$. A functor $\tilde{\Sigma}_A$ is naturally defined, for which

$$\tilde{\Sigma}_A(X) = \text{Map}(\overline{X}, A')$$

(this functor acts from the category $\mathcal{T}$ to the category of sets); the mappings constructed above

$$i_X : A \otimes X \rightarrow \text{Map}(\overline{X}, A')$$

and

$$\tilde{\mu}_X : F(X) \rightarrow \text{Map}(\overline{X}, A')$$

generate mappings of functors (if the functors $\tilde{\Sigma}_A$ and F are also regarded as acting in the category of sets).

Lemma 1. If the condition

$$\tilde{\mu}_{XF}(X) \subset i_X(A \otimes X)$$

is satisfied for $X = P$, then it is satisfied for every space $X \in \mathcal{T}$.

Let $z \in F(X)$, $v = \tilde{\mu}_X z \in \text{Map}(\overline{X}, A')$. We shall show that, under the hypotheses of the lemma, there is a point $a \otimes x \in A \otimes X$ for which $i_X(a \otimes x) = v$. Take a morphism $\sigma : X \rightarrow P$ (such a morphism is, evidently, determined by a pair of morphisms $\psi_1, \psi_2 \in \overline{X}$) and consider the commutative diagram

$$\tilde{\mu}_{PF}(\sigma) = \tilde{\Sigma}_A(\sigma) \tilde{\mu}_X.$$

By the condition of the lemma there is a point

$$a_\sigma \otimes \tau_\sigma \in A \otimes P$$

such that

$$\tilde{\Sigma}_A(\sigma)(v) = \tilde{\mu}_{PF}(\sigma)z = i_P(a_\sigma \otimes \tau_\sigma).$$

Unraveling the definitions of the mappings $\tilde{\Sigma}_A(\sigma)$ and i_P , we arrive at the following assertion, which we shall call ****assertion (*)**** below. Let $\psi_1, \psi_2 \in \overline{X}$; let $\lambda(t_1, t_2)$ be a continuous numerical function of two numerical variables, and let l_λ be the function from $\overline{X}$ defined by the relation

$$l_\lambda(x) = \lambda(\psi_1(x), \psi_2(x)).$$

Then there exist points $\tau = (\tau_1, \tau_2) \in P$ and $a \in A$ (depending on v, ψ_1, ψ_2) such that, for every function λ ,

$$v(l_\lambda) = a \otimes \lambda(\tau_1, \tau_2).$$

Using assertion **(*)**, we first come to the conclusion that if

$$v(\psi_1) = a_1 \otimes v_1 \neq 0, \quad v(\psi_2) = a_2 \otimes v_2 \neq 0,$$

then $a_1 = a_2$ (it is enough to consider the functions $\lambda_1(t_1, t_2) = t_1$ and $\lambda_2(t_1, t_2) = t_2$). In other words, the mapping

$$v : \overline{X} \rightarrow A'$$

has the form

$$v(\varphi) = a_0 \otimes \nu(\varphi),$$

where a_0 is a fixed point of A , and $\nu(\varphi)$ is a numerical functional on $\overline{X}$. It turns out that $\nu(\varphi)$ is a multiplicative linear functional on the space of functions $\overline{X}$. Indeed, applying assertion **(*)** to three functions—an arbitrary function $\lambda(t_1, t_2)$ and the functions $\lambda_1(t_1, t_2) = t_1$, $\lambda_2(t_1, t_2) = t_2$ —we obtain that, for any functions $\psi_1, \psi_2 \in \overline{X}$ and any function $\lambda(t_1, t_2)$, the relation

$$\nu(\lambda(\psi_1, \psi_2)) = \lambda(\nu(\psi_1), \nu(\psi_2))$$

holds. Specializing this relation to the functions

$$\lambda(t_1, t_2) = t_1 t_2$$

and

$$\lambda(t_1, t_2) = \alpha t_1 + \beta t_2,$$

we obtain the multiplicativity and linearity of the functional ν . Next, we endow the space of functions $\overline{X}$ with the compact-open topology and note that, from the continuity condition imposed by us on the functor F , the continuity of the functional $\nu(\varphi)$ follows. It is known ([4], p. 28) that if the space X is completely regular and the space $\overline{X}$ of continuous functions on X is endowed with the compact-open topology, then every continuous multiplicative linear functional ν on $\overline{X}$ is determined by some point $x_0 \in X$ by the formula

$$\nu(\varphi) = \varphi(x_0).$$

Thus we have

$$v(\varphi) = a_0 \otimes \varphi(x_0),$$

i.e.

$$v = i_X(a_0 \otimes x_0).$$

The assertion of the lemma is proved.

Let us now consider the mapping of the set $DF(A)$ into the set $D_R F'(A)$ appearing in the assertion of the theorem. The one-to-one property

of this mapping follows from the possibility, mentioned above, of reconstructing the mapping of functors $\lambda : F \rightarrow \Sigma_A$ from the mapping $\lambda_E : F(E) \rightarrow \Sigma_A(E)$. It remains for us, therefore, to prove that the mapping $DF(A)$ into $D_{\mathcal{R}} F(A)$ is onto. For this, first note that if an element $D_{\mathcal{R}} F(A)$ is determined by a pair of morphisms $\mu : F(E) \rightarrow A \otimes E$ and $\rho : F(P) \rightarrow A \otimes P$, then one may assert that $\mu_P = i_P \rho$, i.e. Lemma 1 is applicable to the morphism μ . By Lemma 1, $\mu_{XF}(X) \subset i_X(A \otimes X)$ for every $X \in \mathcal{T}$, and hence one may define the mapping $\lambda_X = i_X^{-1} \mu_{XF}(X) \rightarrow A \otimes X$, which, however, a priori need not be a morphism. It turns out, however, that from the compatibility of the mappings λ_X and the continuity of the mapping λ_E one can derive the continuity of all the mappings λ_X .

Remark. Theorem 1 may be formulated differently, as a theorem on the independence of the dual functor from the choice of the category with respect to which the dual functor is defined.

If one assumes that the category $\mathcal{T}$ contains T_0 -spaces with a finite number of points, then one can prove an analogue of Theorem 1 in which the spaces E and P are replaced by still simpler spaces (T_0 -spaces with 2, 3, and 4 points). Consider the category $\mathcal{T}$, consisting of T_0 -spaces with a distinguished point 0 and satisfying the following conditions: a) the distinguished point of each space $X \in \mathcal{T}$ is closed; b) if L is such a subset of a space $X \in \mathcal{T}$ that the system V_L

of open sets containing at least one point of L is centered, then the intersection of all sets of the system V_L is nonempty. Suppose also that the category $\mathcal{T}$ contains, in particular, the following spaces: the connected two-point space I (the space with 2 points 0 and 1, the first of which is closed and the second open); the space with 4 points $Q = \{0; 1; 2; 3\}$ with open sets $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{3\}$; the space with 3 points $T = \{0, 1, 2\}$ with open sets $\{1, 2\}$, $\{0, 1\}$, $\{1\}$.

Let F be a functor acting from the category $\mathcal{T}$ to the category of all topological spaces $\mathcal{C}$. Denote by $\mathcal{L}$ the subcategory of the category $\mathcal{T}$ consisting of the three spaces I, Q, T and all morphisms $Q \rightarrow I, T \rightarrow I$. Introduce the set $D_{\mathcal{L}}F(A)$ of mappings of the functor F , considered only on the category $\mathcal{L}$, into the functor Σ_A .

Theorem 2. *If the functor F is continuous, then the natural mapping of the set $DF(A)$ into the set $D_{\mathcal{L}}F(A)$ is a one-to-one mapping onto $D_{\mathcal{L}}F(A)$.*

The proof of Theorem 2 is to a considerable extent analogous to the proof of Theorem 1. Here the role of the line E^1 in the definition of the spaces $\bar{X}$ and A' is played by the connected two-point space I ; the space $\bar{X}$ is naturally realized as the collection of open sets in $X \setminus 0$. The role of the assertion used above from ⁽⁴⁾ is played by

Lemma 2. *Let $\varphi \in \text{Hom}(\bar{X}, A')$. In order that the morphism φ belong to $i_X(A \otimes X)$, it is sufficient that for every pair of sets $M_1, M_2 \in \bar{X}$ one of the following three conditions be fulfilled:*

- either 1) $\varphi(M_1 \cup M_2) = \varphi(M_1) = \varphi(M_2) = \varphi(M_1 \cap M_2)$;
- or 2) $\varphi(M_1 \cup M_2) = \varphi(M_1)$; $\varphi(M_2) = \varphi(M_1 \cap M_2) = 0$;
- or 3) $\varphi(M_1 \cup M_2) = \varphi(M_2)$; $\varphi(M_1) = \varphi(M_1 \cap M_2) = 0$.

The proof of the lemma proceeds according to the following plan. We shall call a set $M \in \bar{X}$ nonzero if $\varphi(M) \neq 0$; we shall call a point $x \in X$ nonzero if every set $M \in \bar{X}$ containing it is nonzero. First of all we prove by contradiction that every nonzero set contains a nonzero point. Next, applying condition b), imposed on the category $\mathcal{T}$, to the set L of all nonzero points, we ascertain that there exists a nonzero point x_0 contained in every nonzero set. To complete the proof, we observe that $\varphi = i_X(x_0 \otimes a)$, where a is the common value of φ on all nonzero sets.

2. Consider a D -category $\mathcal{K}$ satisfying the following conditions: 1) in the category $\mathcal{K}$ the tensor product coincides with the direct product (i.e., there exists an isomorphism of functors of two variables $X \otimes Y$ and $X \times Y$); 2) for an arbitrary spectrum $\{X_\lambda\}$ of objects of the category $\mathcal{K}$, from the existence of the inductive limit $\varinjlim \hat{X}_\lambda$ of the sets $\hat{X}_\lambda$ there follows the existence of the limit $\varinjlim X_\lambda$; 3) in the category $\mathcal{K}$ there is an object J

such that for every object $X \in \mathcal{K}$ the set of morphisms of the object X into the object J is nonempty.

Theorem 3. *For any functor F in a category satisfying the requirements listed above, the dual functor DF is a composition $\Sigma_A \Omega_B$ of the functors Σ_A and Ω_B .*

Proof. By definition,

$$DF(X) = H(F, \Sigma_X) = \varprojlim \{N_\lambda, \Pi'_\lambda\}_{\lambda \in \Lambda},$$

where Λ is the set of morphisms of the category $\mathcal{K}$,

$$N_\lambda = H(F(Y), X \otimes Z), \quad \text{if } \lambda \in \text{Hom}(Y, Z).$$

Let us note that

$$N_\lambda = H(F(Y), X) \times H(F(Y); Z)$$

(this follows from the coincidence of the tensor product with the direct product and from the relation, valid in any D -category,

$$H(A, B \times C) = H(A, B) \times H(A, C)).$$

Therefore $DF(X)$ also decomposes into a product of two factors; it is not difficult to verify that one of them is isomorphic to $DF(I)$, where I is the integral object, and the other to $H(F, T_X)$, where T_X is the functor assigning to each object of the category the fixed object X , and to each morphism the identity morphism. To complete the proof it remains only to verify that $H(F, T_X) = \Omega_B(X)$, where B is the inductive limit of the spectrum $\{F(Y), R_Y^{Y'}\}_{Y \in K}$, in which $R_Y^{Y'}$ is the set of morphisms of the type $F(d)$; the existence of the limit B follows from conditions 2 and 3.

The conditions of Theorem 3 are satisfied by the categories of sets, functionally Hausdorff k -spaces (3), and partially ordered sets (as morphisms one takes, respectively, arbitrary mappings, continuous mappings, and mappings preserving order). In the categories just listed, with the aid of Theorem 3 it can be shown that the stock of reflexive functors is exhausted by functors of the form Σ_A and Ω_A (and, consequently, the relation $D^3 = D$ is not satisfied).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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