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Abstract

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MATHEMATICS

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APPROXIMATION OF CONTINUOUS FUNCTIONS BY HOLOMORPHIC FUNCTIONS ON JORDAN ARCS IN C^n

(Presented by Academician M. A. Lavrent'ev on 14 VI 1965)

In this paper we study the possibility of uniform approximation of continuous functions on simple Jordan arcs in the space C^n of complex variables $z = (z_1, \dots, z_n)$ by functions holomorphic on these arcs. By $C(X)$ we denote the algebra of all continuous complex-valued functions on a compact set X , and by $A(X)$, $X \subset C^n$, the algebra of all uniform limits of functions holomorphic on X .

1. **Theorem 1.** *Let γ be a simple Jordan arc in C^2 such that its projection $z_1(\gamma)$ is a nowhere dense subset of C^1 . Then $C(\gamma) = A(\gamma)$.*

Proof. 1) We first show that for every polynomial $p(z_1)$ and every positive integer m , the function

$$\sqrt[m]{p(z_1)}$$

belongs to $A(\gamma)$.

For this it is enough to prove that

$$\sqrt[m]{z_1 - a} \in A(\gamma)$$

for any point $a \in C^1$. Let $a \in z_1(\gamma)$, and let a_ν , $\nu = 1, 2$, be such that $a_\nu \in C^1 \setminus z_1(\gamma)$ and $a_\nu \rightarrow a$ as $\nu \rightarrow \infty$. For each ν , in view of the connectedness of γ , one can construct a function holomorphic on γ ,

$$\sqrt[m]{z_1 - a_\nu}$$

(see (1)). Let $\gamma_a = \{z \in \gamma : z_1 = a\}$, and let E_δ be the subset of γ consisting of a finite number of connected components, containing the set $\{z \in \gamma : |z_1 - a| \geq \delta > 0\}$ and not intersecting γ_a . Then from the sequence of functions $\sqrt[m]{z_1 - a_\nu}$, $\nu = 1, 2, \dots$, one can choose a subsequence converging uniformly on E_δ . Taking $\delta = 1/n$ and letting $n \rightarrow \infty$, by a diagonal process we extract a subsequence

$$\sqrt[m]{z_1 - a_{\nu_k}}, \quad k = 1, 2, \dots,$$

which converges uniformly on any compact subsets of $\gamma \setminus \gamma_a$.

Let $\varepsilon > 0$ be given. Choose a neighborhood U of the set γ_a in C^n such that $|z_1 - a_\nu| < \varepsilon^m$ if $z \in U$ and $a_\nu \in z_1(U)$. Choose k_1 so that $a_{\nu_k} \in z_1(U)$, $k > k_1$, and $\delta > 0$ so that $\gamma \setminus U \subset E_\delta$. In view of the uniform convergence of the sequence $\{\sqrt[m]{z_1 - a_{\nu_k}}\}$ on E_δ , there is k_2 such that

$$\left| \sqrt[m]{z_1 - a_{\nu_{k+1}}} - \sqrt[m]{z_1 - a_{\nu_k}} \right| < \varepsilon$$

for $k > k_2$. Then for $k > \max(k_1, k_2)$ we obtain, uniformly on γ ,

$$\left| \sqrt[m]{z_1 - a_{\nu_{k+1}}} - \sqrt[m]{z_1 - a_{\nu_k}} \right| < 2\varepsilon,$$

whence it follows that

$$\sqrt[m]{z_1 - a} \in A(\gamma),$$

and therefore

$$\sqrt[m]{p(z_1)} \in A(\gamma).$$

Consider the algebra generated by functions of the form

$$\sqrt[m]{p(z_1)},$$

where p is an arbitrary polynomial and m is any natural number. In a completely analogous way it is proved that roots of any degree of functions of this algebra belong to $A(\gamma)$. From the roots obtained we construct a new algebra, etc. The union of all these algebras forms a subalgebra $A_1 \subset A(\gamma)$, which, together with each f , also contains

$$\sqrt[m]{f}$$

for any natural m . By a diagonal process, as was done above, this property can be extended to the uniform closure $\bar{A}_1$ of the algebra A_1 on γ .

- 2) Let V and E be connected compact subsets of γ such that $|z_1 - a| < \alpha$ for $z_1 \in z_1(V)$ and $|z_1 - a| > \alpha$ for $z_1 \in z_1(E)$. Put

$$f_m = \alpha \sqrt[m]{\left(\frac{z_1 - a}{\alpha}\right)^m + 1};$$

we have

$$\lim_{m \rightarrow \infty} |f_m| = \max(|z_1 - a|, \alpha)$$

and

$$\lim_{m \rightarrow \infty} f_m(z) = \alpha$$

for $z \in V$, in view of the connectedness of V . Let

$$f_{mN} = \sqrt[N]{f_m - \alpha};$$

then

$$\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} f_{mN}(z) = 0,$$

if $z \in V$. Since the variation of the argument $\Delta_E \text{Arg}(f_m - \alpha)$ on the connected compact set E does not exceed $\Delta_E \text{Arg}(z_1 - a) + 2\pi$, choosing the necessary branches, we may assume that

$$\lim_{N \rightarrow \infty} \lim_{m \rightarrow \infty} f_{mN}(z) = 1,$$

if $z \in E$. Therefore, for any $\delta' > 0$ one can choose m and N so that $|f_{mN}| < 1 + \delta'$, $|f_{mN}(z) - 1| < \delta'$, $z \in E$, and $|f_{mN}(z)| < \delta'$, $z \in V$.

Let now U_a be a neighborhood of γ_a in γ , consisting of a finite number of connected components $U_{a\nu}$, $\nu = 1, \dots, n$, containing the set $\{z \in \gamma : |z_1 - a| \leq \alpha, \alpha > 0\}$ and contained in $\{z \in \gamma : |z_1 - a| < \alpha + \beta, \beta > 0\}$. Let E_a be a subset of γ , consisting of a finite number of connected compact sets $E_{a\mu}$, such that

$$\{z \in \gamma : |z_1 - a| \geq \alpha + 2\beta\} \subset E$$

and

$$E \subset \{z \in \gamma : |z_1 - a| > \alpha + \beta\}.$$

In view of the local connectedness of γ , such sets can be constructed for any $\alpha, \beta > 0$. For each pair $U_{a\nu}, E_{a\mu}$ and any $\delta' > 0$ we can now construct a function $g_{\mu\nu}$ such that $|g_{\mu\nu}| < 1$, $|g_{\mu\nu}(z) - 1| < \delta'$, $z \in E_{a\mu}$, and $|g_{\mu\nu}(z)| < \delta'$, $z \in \overline{U}_{a\nu}$.

Let

$$g_\nu = \prod_{\mu} g_{\mu\nu}$$

and

$$g = 1 - \sqrt[n]{(1 - g_1) \cdots (1 - g_n)}.$$

Then, if $\delta'' > 0$, δ' can be chosen so that $|g| \leq 1$, $|g(z) - 1| < \delta''$, if $z \in E$, and $|g(z)| \leq \delta''$, if $z \in \overline{U}_a$. Let $1 > \delta > 0$, and let φ be a fractional-linear transformation of the unit disk onto itself such that $\varphi(0) = -1 + \delta$ and $\varphi(1) = 1$. Choose δ'' so that $|\varphi(\lambda) - 1| < \delta^4$, if $|\lambda - 1| < \delta''$, and $|\varphi(\lambda) + 1| < 2\delta^4$, if $|\lambda| < \delta''$, and put

$$h_a = \sqrt[4]{\frac{1}{2}(1 - \varphi \circ g)};$$

according to 1), $h_a \in A(\gamma)$. Moreover:

$$|h_a| \leq 1, \quad |\arg h_a| \leq \pi/8,$$

$$|h_a(z) - 1| < \delta, \quad \text{if } z \in \overline{U}_a,$$

and

$$|h_a(z)| < \delta, \quad \text{if } z \in E_a.$$

Let now $f \in C(\gamma)$ and $\varepsilon > 0$. By the theorem of M. A. Lavrentiev ⁽²⁾, for every $a \in z_1(\gamma)$ there is a polynomial

$$p_a(z) \equiv p_a(z_1)$$

such that $|f(z) - p_a(z)| < \varepsilon$, if $z \in \gamma_a$. Choose α and β so that $|f - p_a| < \varepsilon$ on $\gamma \setminus E_a$. From the covering of γ by the sets U_a select a finite subcover: $U_1, \dots, U_k$. Let $p_1, \dots, p_k$ be the corresponding polynomials and $h_1, \dots, h_k$ the functions constructed above for δ and for the pairs U_i, E_i ; we assume that $0 < \delta < 1/2$. Put

$$e_i = h_i(h_1 + \dots + h_k)^{-1}.$$

Then

$$e_1 + \dots + e_k \equiv 1$$

on γ , and

$$|\arg e_i| < \pi/4, \quad i = 1, \dots, k.$$

Moreover,

$$|e_i(z)(f(z) - p_i(z))| < \varepsilon, \quad z \in \gamma \setminus E_i,$$

and

$$|e_i(z)(f(z) - p_i(z))| < 2 \max_{\gamma} |f - p_i| \cdot \delta, \quad z \in E_i.$$

Let $z \in \gamma$, and let $i_1, \dots, i_s$ be all indices such that $z \notin E_{i_\nu}$, $\nu = 1, \dots, s$. Then

$$\begin{aligned} \left| f(z) - \sum_{i=1}^k e_i(z)p_i(z) \right| &= \left| \sum_{i=1}^k e_i(z)(f(z) - p_i(z)) \right| \leq \\ &\leq \left| \sum_{\nu=1}^s e_{i_\nu}(z)(f(z) - p_{i_\nu}(z)) \right| + \left| \sum_{\nu=s+1}^k e_{i_\nu}(z)(f(z) - p_{i_\nu}(z)) \right| < \\ &< \varepsilon \sum_{\nu=1}^k |e_{i_\nu}(z)| + \left(2k \max_i \max_{\gamma} |f - p_i| \right) \delta. \end{aligned}$$

Since

$$\sum_{\nu=1}^k |e_{i_\nu}(z)| \leq \sqrt{2},$$

with a proper choice of δ we obtain uniformly on all of γ

$$\left| f - \sum_{i=1}^k e_i p_i \right| < \varepsilon.$$

Since

$$\left(\sum_{i=1}^k e_i p_i \right) \in A(\gamma),$$

the theorem is proved.

2. Theorem 2. *Let γ be a simple Jordan arc in C^2 . Then, for every function j holomorphic on γ , either the set $j(\gamma)$ contains interior points, or $C(\gamma) = A(\gamma)$.*

In the proof of Theorem 1 we used Lavrentiev's theorem on approximation by polynomials on plane compact sets. The proof of Theorem 2 rests on the following generalization of S. N. Mergelyan's theorem ⁽³⁾:

Theorem 3. *Let O be an open subset of C^n and S a one-dimensional analytic set in O . Let a compact set $X \subset S$ not divide the irreducible components of this set. Then: 1) the set X is polynomially convex in C^n ; 2) every function continuous on X and holomorphic at the points of X that are interior relative to S is uniformly approximable on X by polynomials in $z_1, \dots, z_n$.*

Theorem 3 is proved using results of Rossi ⁽⁴⁾ and Bishop ⁽⁵⁾. It is interesting to compare Theorem 2 with the known examples of Wermer ⁽⁶⁾ and Rudin ⁽⁷⁾ of Jordan arcs in C^3 and C^2 on which not every continuous function is approximable by polynomials. However, in these examples, for every polynomial $p(z)$, the plane set $p(\gamma)$ contains no interior points.

3. **Definition.** Let A be a multiplicative group. We shall say that A is **closed under extraction of (square) roots** if, together with every $f \in A$, A also contains an element $\sqrt{f}$ such that $(\sqrt{f})^2 = f$. Let X be an arbitrary compact set, and let A_0 be a subset of $C(X)$. By the **algebraic extension** of the set A_0 we mean the minimal subalgebra of $C(X)$ that contains A_0 , is closed with respect to uniform convergence on X , and is closed with respect to the operation of extracting square roots.

In the first part of the proof of Theorem 1, essentially the following was proved.

Theorem 4. *Let X be a locally connected compact set, and let A be some family of functions continuous on X , closed with respect to extraction of roots. Then $\overline{A}$ (the uniform closure of A on X) is also closed with respect to this operation.*

In the second part of the proof of Theorem 1 one can get by with roots of powers of the form 2^k , where k is a natural number. In doing so, only the following facts are essentially used: 1) the arc γ is locally connected; 2) there exists a family A_0 of functions from $A(\gamma)$ (namely, the functions z_1), whose algebraic extension lies in $A = A(\gamma)$; 3) on every set γ_a , where all functions from A_0 are constant, $C(\gamma_a) = A|_{\gamma_a}$ ($A|_{\gamma_a}$ denotes the restriction of A to γ_a).

Thus the following holds.

Theorem 5. *Let X be a locally connected compact set; let A and B , $A \subset B$, be algebras of continuous functions on X . Suppose that in A there exists a subset A_0 such that: 1) the algebraic extension of A_0 lies in A ; 2) on every subset $X_a \subset X$ on which all functions from A_0 are constant, $\overline{B|_{X_a}} = \overline{A|_{X_a}}$. Then $\overline{A} = \overline{B}$.*

If A_0 separates the points of X , then condition 2) is automatically satisfied, and therefore the following holds.

Theorem 5'. *Let A be a separating algebra of continuous functions with identity on a locally connected compact set X , closed with respect to extraction of square*

roots. Then $\overline{A} = C(X)$.

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