

On Mutually Adjoint Boundary Value Problems for a System of Three First-Order Composite-Type Equations

A. Dzhuraev

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Abstract

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MATHEMATICS

A. DZHURAEV

ON MUTUALLY ADJOINT BOUNDARY-VALUE PROBLEMS FOR A SYSTEM OF THREE FIRST-ORDER EQUATIONS OF COMPOSITE TYPE

(Presented by Academician I. N. Vekua, 31 1966)

A system of first-order partial differential equations

$$U_x - AU_y - BU = 0, \quad (1)$$

where A, B are real square matrices of order three, and $U = (u, v, w)$ is the unknown three-component vector, is called a **system of composite type in the domain $\widetilde{G}$** if the equation $\det(A - \lambda E) = 0$, where E is the identity matrix, has in the domain $\widetilde{G}$ roots $\lambda_1(z), \lambda_0(z), \overline{\lambda_0(z)}$ such that $\operatorname{Im} \lambda_1(z) \equiv 0$ in $\widetilde{G}$, while $\operatorname{Im} \lambda_0(z)$ does not vanish for any value $z \in \widetilde{G}$. Assuming the matrix A to be continuously differentiable in $\widetilde{G}$ and taking, for definiteness, $\operatorname{Im} \lambda_0(z) < 0$ in $\widetilde{G}$, system (1) can be reduced to the form

$$\begin{aligned} f_x - \lambda_1 f_y &= A_0 f + \operatorname{Re}(B_0 \varphi), \\ \varphi_x - \lambda_0 \varphi_y &= A_1 f + B_1 \varphi + C_1 \bar{\varphi}, \end{aligned} \quad (2)$$

where $f = \chi^1 U$, $\varphi = \chi^0 U$, and χ^j is a linearly independent solution of the equation $(A' - \lambda E)\chi = 0$, corresponding to $\lambda = \lambda_j(z)$, $j = 0, 1$. Let $\xi(x, y) = \text{const}$ be an integral of the equation $dy + \lambda_1(x, y) dx = 0$. Then, making a transformation of the independent variables $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ with positive Jacobian in $\widetilde{G}$, system (2) can be reduced to the canonical form:

$$\begin{aligned} f_\eta &= A_0 f + \operatorname{Re}(B_0 \varphi), \\ \varphi_\xi - \mu_0(\xi) \varphi_\eta &= A_1 f + B_1 \varphi + C_1 \bar{\varphi}, \end{aligned} \quad (3)$$

where $\mu_0(\xi), B_0(\xi), B_1(\xi), C_1(\xi)$ are fully determined complex-valued functions in the domain G , the image of the domain $\widetilde{G}$, with $\text{Im } \mu_0(\xi) < 0$ everywhere in G , and $A_0(\xi)$ is a real-valued function in G .

Regarding the domain G , we shall assume that it is a bounded simply connected domain whose boundary Γ is a simple closed Lyapunov curve, moreover such that the contour Γ has tangents $\xi = \xi_0, \xi = \xi^0$ at the points M and N , respectively, and every straight line $\xi = \xi^*$ with $\xi_0 < \xi^* < \xi^0$ intersects Γ in exactly two points, while the straight lines $\xi = \xi^*$ with $\xi^* < \xi_0$ and $\xi^* > \xi^0$ have no common points with Γ . Then the points M and N divide the contour Γ into two arcs: γ and $\Gamma - \gamma$. We pose the following problem:

Problem A. Find continuously differentiable in G solutions $W = (f, \varphi)$ of system (3), continuously extendable to the contour Γ and satisfying the boundary conditions:

$$B_0(W) \equiv \text{Re } \Delta(t_0)\varphi(t_0) = h^0(t_0), \quad t_0 \in \gamma; \quad (4)$$

$$B_1(W) \equiv f(t_0) + \text{Re } d(t_0)\varphi(t_0) = h^1(t), \quad t_0 \in \gamma; \quad (5)$$

$$B_2(W) \equiv a_0(t_0)f(t_0) + \text{Re } a^0(t_0)\varphi(t_0) = h_0(t_0), \quad t_0 \in \Gamma - \gamma, \quad (6)$$

where $a_0(t)$ is real-valued, and $\Delta(t), d(t), a^0(t)$ are complex-valued functions, Hölder-continuous on the corresponding parts of the contour Γ , with-

where

$$\lim_{\substack{t \rightarrow C \\ t \in \Gamma - \gamma}} a_0(t) = \lim_{\substack{t \rightarrow C \\ t \in \gamma}} d(t) = 0, \quad \lim_{\substack{t \rightarrow C \\ t \in \Gamma - \gamma}} a^0(t) = \lim_{\substack{t \rightarrow C \\ t \in \gamma}} \Delta(t) \neq 0, \quad C = M, N.$$

We shall assume that the following normality conditions are satisfied:

$$\Delta(t) \neq 0 \quad \text{for } t \in \gamma; \quad a^0(t) \neq 0 \quad \text{for } t \in \Gamma - \gamma. \quad (\text{N})$$

The problem posed is so general that the general* problem

$$\begin{aligned} a_0(t_0)f(t_0) + \text{Re } a^0(t_0)\varphi(t_0) &= h_0(t_0), \quad t_0 \in \Gamma, \\ a_1(t_0)f(t_0) + \text{Re } a^1(t_0)\varphi(t_0) &= h_1(t_0), \quad t_0 \in \gamma, \end{aligned} \quad (7)$$

is easily reduced to it, if

$$\Delta(t) = a^0(t)a_1(t) - a_0(t)a^1(t) \neq 0 \quad \text{for } t \in \gamma.$$

It is convenient to write system (3) in complex form ⁽¹⁾

$$D(W) \equiv W_\xi - QW_\zeta - A_0W - B_0\overline{W} = 0, \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\mu_0 + i}{\mu_0 - i} \end{pmatrix}. \quad (8)$$

Along with system (8), consider the formally adjoint system

$$D_*(V) \equiv -V_\xi + (QV)_\xi - A'_0V - \overline{B'_0}\overline{V} = 0, \quad (9)$$

and for it pose the following problem (adjoint to problem A):

Problem A^0_* . Find continuously differentiable in G solutions $V = (ig, \psi)$ (g real-valued) of system (9), continuously extendable to the contour Γ and satisfying the following homogeneous conditions:

$$B^*_0(V) \equiv -\operatorname{Im} \frac{d(t_0)}{\Delta(t_0)} \xi'(s_0)g(t_0) + \operatorname{Im} \frac{t'(s_0) + q(t_0)\overline{t'(s_0)}}{2i\Delta(t_0)} \psi(t_0) = 0, \quad t_0 \in \gamma; \quad (10)$$

$$B^*_1(V) \equiv \xi'(s_0)g(t_0) - a_0(t_0) \operatorname{Im} \frac{t'(s_0) + q(t_0)\overline{t'(s_0)}}{2ia^0(t_0)} \psi(t_0) = 0, \quad t_0 \in \Gamma - \gamma; \quad (11)$$

$$B^*_2(V) \equiv \operatorname{Im} \frac{t'(s_0) + q(t_0)\overline{t'(s_0)}}{2ia^0(t_0)} \psi(t_0) = 0, \quad t_0 \in \Gamma - \gamma; \quad (12)$$

$$q(t) = (\mu_0(t) + i)/(\mu_0(t) - i).$$

Let $G_1(t)$ and $G_2(t)$ be certain nonsingular complex-valued square matrices, prescribed respectively on γ and $\Gamma - \gamma$. Then from Green's identity

$$\operatorname{Re} \iint_G \{VD(W) - WD_*(V)\} d\xi d\eta = \operatorname{Re} \frac{1}{2i} \int_\Gamma \overline{W}(t'(s) + Q\overline{t'(s)})V ds$$

it is not difficult to derive that

$$\operatorname{Re} \iint_G \{VD(W) - WD_*(V)\} d\xi d\eta =$$

$$\begin{aligned}
&= \sum_{k=1}^2 \int_{\gamma_k} \left\{ \operatorname{Re}(G_{kW}) \operatorname{Re} \left(\frac{1}{2i} G_{k'}^{-1}(t) [t'(s) + Q\overline{t'(s)}] V \right) \right. \\
&\quad \left. - \operatorname{Im}(G_{kW}) \cdot \operatorname{Im} \left(\frac{1}{2i} G_{k'}^{-1}(t) [t'(s) + Q\overline{t'(s)}] V \right) \right\} ds, \quad \gamma_1 = \gamma, \quad \gamma_2 = \Gamma - \gamma.
\end{aligned} \tag{13}$$

If now, as the matrices $G_1(t)$, $G_2(t)$, one chooses the following nonsingular matrices by virtue of condition (N):

$$G_1(t) = \begin{pmatrix} 0 & \Delta(t) \\ 1 & d(t) \end{pmatrix}, \quad G_2(t) = \begin{pmatrix} i & 0 \\ a_0(t) & a^0(t) \end{pmatrix},$$

and assumes that $W = (f, \varphi)$ is a solution of problem A, while $V = (ig, \psi)$ is a solution of problem A_*^0 , then from identity (13) it necessarily follows that

$$\begin{aligned}
&\int_{\gamma} h^0(t) \left[\frac{t'(s) + q(t)\overline{t'(s)}}{2i\Delta(t)} \psi(t) - \frac{d(t)}{\Delta(t)} \xi'(s)g(t) \right] ds + \int_{\gamma} h'(t)\xi'(s)g(t) ds + \\
&\quad + \int_{\Gamma-\gamma} h_0(t) \frac{t'(s) + q(t)\overline{t'(s)}}{2ia^0(t)} \psi(t) ds = 0.
\end{aligned} \tag{R}$$

* Problem (7) was posed by the author in paper ⁽²⁾.

Thus, we have

Lemma 1. *If the homogeneous problem A_*^0 has a nonzero solution, then for the solvability of problem A it is necessary that the right-hand sides satisfy equality (R).*

We shall prove that conditions (R) are also sufficient for the solvability of problem A. To this end, for simplicity, we shall carry out the proof for the case when in the equations the matrices $A_0 = B_0 \equiv 0$. In this case $f \equiv f_0(\xi)$, where $f_0(\xi)$ is an arbitrary real function of the variable ξ , continuous on the closed interval $[\xi_0, \xi^0]$, and $\varphi(\xi)$ is a solution of the Beltrami equation $\varphi_{\xi} - q(\xi)\varphi_{\bar{\xi}} = 0$.

Using the integral representation

$$\begin{aligned}
\varphi(\xi) &= \frac{1}{\pi i} \int_{\Gamma} Z(t, \xi) \mu(s) \times \\
&\quad \times (dt + q(t) \overline{dt}) + iC,
\end{aligned}$$

where $Z(t, \xi)$ is the Cauchy kernel of the Beltrami equation, we reduce problem A to the equivalent functional equation

$$\begin{aligned} K(\mu) &\equiv \operatorname{Re} a^*(s_0)\mu(s_0) + \operatorname{Re} b^*(s_0)\mu[\alpha(s_0)] + \\ &+ \int_{\Gamma} \operatorname{Re} \left\{ \frac{a^*(s_0)}{\pi i} Z[t(s), t(s_0)][t'(s) + q(t)\overline{t'(s)}] \right\} \mu(s) ds + \\ &+ \int_{\Gamma} \operatorname{Re} \left\{ \frac{b^*(s_0)}{\pi i} Z[t(s), t(\alpha(s_0))][t'(s) + q(t)\overline{t'(s)}] \right\} \mu(s) ds = H(s_0), \end{aligned} \quad (14)$$

where $s^* = \alpha(s)$ is a differentiable function determined from the equation $\xi(s) = \xi(s^*)$, $s \in \gamma$, $s^* \in \Gamma - \gamma$;

$$a^*(s) = \begin{cases} \Delta(t(s)), & s \in \gamma, \\ a^0(t(s)), & s \in \Gamma - \gamma, \end{cases} \quad b^*(s) = \begin{cases} 0, & s \in \gamma, \\ -a_0(t(s)) d[t(\alpha(s))], & s \in \Gamma - \gamma; \end{cases}$$

$$H(s) = h^*(s) + C \operatorname{Im}[a^*(s) + b^*(s)];$$

$$h^*(s) = \begin{cases} h^0(t(s)), & s \in \gamma, \\ h_0(t(s)) - a_0(t(s)) h^1[t(\alpha(s))], & s \in \Gamma - \gamma. \end{cases}$$

Equation (14) is a singular integral equation with shift. Let now $V = (ig, \psi)$ be a nontrivial solution of problem A_*^0 . Then $g \equiv g_0(\xi)$, where $g_0(\xi)$ is an arbitrary real function of the variable ξ , continuous on the interval $[\xi_0, \xi^0]$, and $\psi(\xi)$ is a solution of the equation $\psi_{\xi} - (q\psi)_{\bar{\xi}} = 0$. Considering the real continuous function

$$v(s) = \begin{cases} -\frac{d(t)}{\Delta(t)} \xi'(s) g_0(\xi) + \frac{t'(s) + q(t)\overline{t'(s)}}{2i\Delta(t)} \psi(t), & s \in \gamma, \\ \frac{t'(s) + q(t)\overline{t'(s)}}{2ia^0(t)} \psi(t), & s \in \Gamma - \gamma, \end{cases} \quad (15)$$

and taking condition (11) into account, we shall have

$$\psi(t) = \frac{2ia^*(s)v(s) - 2ib^*[\alpha(s)]v[\alpha(s)]}{t'(s) + q(t)\overline{t'(s)}}, \quad t \in \Gamma. \quad (15')$$

Hence, using the necessary and sufficient condition (1) that $\psi(t)$ be the boundary value of a solution, continuous in $G + \Gamma$, of the equation $\psi_{\xi} - (q\psi)_{\bar{\xi}} = 0$, for the determination of $v(s)$ we obtain the following homogeneous integral equation with shift, adjoint to (14):

$$\begin{aligned} K'(v) &\equiv \operatorname{Re} a^*(s_0)v(s_0) - \operatorname{Re} b^*[\alpha(s_0)]\alpha'(s_0)v[\alpha(s_0)] + \\ &+ \int_{\Gamma} \operatorname{Re} \left\{ \frac{a^*(s_0)}{\pi i} Z[t(s_0), t(s)] + \right. \end{aligned}$$

$$+\frac{b^*(s_0)}{\pi i}Z[t(s_0), t(\alpha(s))]\Big\}[t'(s) + q(t)\overline{t'(s)}]v(s)ds = 0. \quad (16)$$

When condition (N) is fulfilled, equation (14) is normally solvable; for the solvability of equation (14) it is necessary and sufficient that the equalities

$$\int_{\Gamma} H(s)v_j(s)ds = 0, \quad j = 1, 2, \dots, k', \quad (17)$$

hold, where $v_1(s), \dots, v_{k'}(s)$ is a complete system of linearly independent solutions

equation (16). The characteristic part K^0 of the operator K has the form

$$K^0(\mu) \equiv \operatorname{Re} a^*(s_0)\mu(s_0) + \operatorname{Re} b^*(s_0)\mu(\alpha(s_0)) + \\ + \frac{\operatorname{Im} a^*(s_0)}{\pi} \int_{\Gamma} \frac{\mu(s)ds}{s-s_0} + \frac{\operatorname{Im} b^*(s_0)}{\pi} \int_{\Gamma} \frac{\mu(s)ds}{s-\alpha(s_0)}.$$

Let us compute the index of the operator K . To this end we consider the singular operator S of normal type (see condition (N)),

$$S(\mu) \equiv \operatorname{Re} \frac{1}{a^*(s_0)}\mu(s_0) + \frac{1}{\pi} \operatorname{Im} \frac{1}{a^*(s_0)} \int_{\Gamma} \frac{\mu(s)ds}{s-s_0}.$$

The characteristic part K_1^0 of the operator SK^0 has the form

$$K_1^0(\mu) \equiv \mu(s_0) + \operatorname{Re} \frac{b^*(s_0)}{a^*(s_0)}\mu(\alpha(s_0)) - \frac{1}{\pi} \operatorname{Im} \frac{b^*(s_0)}{a^*(s_0)} \int_{\Gamma} \frac{\mu(s)ds}{s-\alpha(s_0)}.$$

We shall prove that $\operatorname{Ind} K_1^0 = 0$. For this, noting that $b^*(s) = 0$ for $s \in \gamma_1$, $b^*(\alpha(s)) = 0$ for $s \in \gamma_2$, one easily verifies the validity of the equality

$$K_1^0 = -K_2 \cdot K_2^* + T,$$

where T is a completely continuous operator, and

$$K_2 = S_1 + S_2, \quad K_2^* = S_1 - S_2,$$

where

$$S_1(\mu) \equiv \frac{1}{\pi} \int_{\Gamma} \frac{[\mu(s)ds]}{s-s_0}, \\ S_2(\mu) \equiv \operatorname{Re} \frac{b^*(s_0)}{2ia^*(s_0)}\mu(\alpha(s_0)) - \frac{1}{\pi} \operatorname{Im} \frac{b^*(s_0)}{2ia^*(s_0)} \int_{\Gamma} \frac{\mu(s)ds}{s-\alpha(s_0)}.$$

Therefore,

$$\operatorname{Ind} K_1^0 = \operatorname{Ind}(K_2 \cdot K_2^*).$$

But

$$-S_1 K_2 = K_3 + T_1, \quad -S_1 K_2^* = K_3^* + T_2,$$

where T_j are completely continuous operators,

$$K_3(\mu) \equiv \mu(s_0) - \operatorname{Re} \frac{b^*(s_0)}{2a^*(s_0)} \mu(\alpha(s_0)) + \frac{1}{\pi} \operatorname{Im} \frac{b^*(s_0)}{2a^*(s_0)} \int_{\Gamma} \frac{\mu(s) ds}{s - \alpha(s_0)},$$

$$K_3^*(\mu) \equiv \mu(s_0) + \operatorname{Re} \frac{b^*(s_0)}{2a^*(s_0)} \mu(\alpha(s_0)) - \frac{1}{\pi} \operatorname{Im} \frac{b^*(s_0)}{2a^*(s_0)} \int_{\Gamma} \frac{\mu(s) ds}{s - \alpha(s_0)}.$$

Consequently,

$$\operatorname{Ind}(K_3 \cdot K_3^*) = \operatorname{Ind}(K_2 \cdot K_2^*) = \operatorname{Ind} K_1^0,$$

since $\operatorname{Ind} S_1 = 0$. But it is easy to verify that

$$K_2 \cdot K_2^* = I + T,$$

where I is the identity and T is a completely continuous operator, whence it follows that

$$\operatorname{Ind} K_1^0 = 0.$$

But then

$$\operatorname{Ind} K = -\operatorname{Ind} S = \frac{1}{\pi} \{\arg a^*(s)\}_{\Gamma}.$$

If we now assume that condition (R) is fulfilled, then, by virtue of (11) and (15), it reduces to the equality

$$\int_{\Gamma} h^*(s) \nu(s) ds = 0. \quad (18)$$

If all the numbers

$$\int_{\Gamma} \operatorname{Im}[a^*(s) + b^*(s)] \nu_j(s) ds, \quad j = 1, 2, \dots, k',$$

are equal to zero, then conditions (17) and (18) coincide, and our assertion is proved. If, however, at least one of these numbers is nonzero, then, putting, for example,

$$\int_{\Gamma} \operatorname{Im}[a^*(s) + b^*(s)] \nu_1(s) ds = 1,$$

we have

$$C = - \int_{\Gamma} h^*(s) \nu_1(s) ds,$$

and then condition (17) has the form

$$\int_{\Gamma} h^*(s) \nu_j(s) ds = 0, \quad j = 2, 3, \dots, k',$$

and the sufficiency of condition (R) is completely proved. Moreover, it follows from this that the index of problem A is equal to

$$k - k' + 1 = \text{Ind } K + 1.$$

Thus, if condition (N) is fulfilled, then Theorems 1 and 2 hold.

Theorem 1. *For the solvability of problem A it is necessary and sufficient that condition (R) be fulfilled.*

Theorem 2. *The index of problem A is computed by the formula*

$$\text{Ind } A = 1 + \frac{1}{\pi} \{\arg a^*(t)\}_{\Gamma}.$$

In the general case, problem A reduces to equation (14) with an additional term that is a completely continuous operator, and the results remain unchanged.

Physical-Technical Institute
Academy of Sciences of the Tajik SSR

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Note: Figure translations are in progress. See original paper for figures.

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