

BOUNDARY-VALUE PROBLEMS FOR ABSTRACT ELLIPTIC EQUATIONS

V. A. Trenogin

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Abstract

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MATHEMATICS

V. A. TRENOGIN

BOUNDARY-VALUE PROBLEMS FOR ABSTRACT ELLIPTIC EQUATIONS

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On $(0, T)$ consider the equation

$$\ddot{x}(t) - B(t)x(t) = h(t). \quad (1)$$

Here $x(t)$ is the unknown, and $h(t)$ is a known function, defined on $[0, T]$ with values in a Banach space E . Differentiation is understood in the strong sense. For each $t \in [0, T]$, the operator $B(t)$ is a linear operator in E with domain of definition $D(B(t))$. By a solution of equation (1) we shall mean a function $x(t)$ such that: 1) $x(t)$ is strongly continuous on $[0, T]$ and twice strongly continuously differentiable on $(0, T)$; 2) $x(t) \in D(B(t))$; 3) $x(t)$ turns (1) into an identity for $t \in (0, T)$. For equation (1) we consider the boundary-value problem: find a solution of equation (1) satisfying the boundary conditions

$$\gamma_i(x) \equiv \gamma_{i1}x(0) + \gamma_{i2}x(T) + \gamma_{i3}\dot{x}(0) + \gamma_{i4}\dot{x}(T) = a_i. \quad (2)$$

Here $i = 1, 2$; $a_i \in E$; γ_{ij} are numbers. It is assumed here that $x(t)$ has one-sided derivatives at $t = 0$ or at $t = T$, if these derivatives enter into (2).

The most important are:

A_1 . The first boundary-value problem: $x(0) = a_1$, $x(T) = a_2$.

A_2 . The second boundary-value problem: $\dot{x}(0) = a_1$, $\dot{x}(T) = a_2$.

A_3 . The problem with periodic conditions $x(0) = x(T)$, $\dot{x}(0) = \dot{x}(T)$.

Let us formulate precise conditions on the parameters of equation (1). Suppose the following conditions are satisfied:

I. For each $t \in [0, T]$, $B(t)$ is a closed linear operator in E with domain of definition $D(B(t))$ dense in E , and for all $\lambda \geq 0$ the inequality

$$\|[B(t) + \lambda I]^{-1}\| \leq M/\lambda + 1$$

holds.

- II. Define $B^{1/2}(t)$ (see (1))—a closed linear operator in E with domain of definition $D(B^{1/2}(t))$ dense in E . Let $D(B^{1/2}(t)) \equiv D$ not depend on t . Assume also that for $t, \tau \in [0, T]$ the operator $B^{1/2}(t)B^{-1/2}(\tau)$ is continuously differentiable with respect to t in the uniform operator topology and $\dot{B}^{1/2}(t)B^{-1/2}(\tau)$ satisfies a Hölder condition with exponent $\gamma \in (0, 1]$, i.e., for $t, s, \tau \in [0, T]$,

$$\|B^{1/2}(t)B^{-1/2}(\tau)\| \leq K, \quad \|[\dot{B}^{1/2}(t) - \dot{B}^{1/2}(s)]B^{-1/2}(\tau)\| \leq L|t - s|^\gamma.$$

- III. $h(t)$ satisfies a Hölder condition

$$\|h(t) - h(s)\| \leq C|t - s|^\eta, \quad \eta \in (0, 1].$$

Let us now observe that from condition I it follows that the operator $-B^{1/2}(t)$ for each $t \in [0, T]$ is the infinitesimal generator of an analytic semigroup $\exp\{-\xi B^{1/2}(t)\}$ with exponential decrease

$$\|\exp\{-\xi B^{1/2}(t)\}\| \leq Ne^{-\beta\xi}, \quad \xi \geq 0$$

(see (2, 3)). Denote by $U(t, s)$

($0 \leq s \leq t \leq T$) the operator solution of the Cauchy problem $\dot{X} + B^{1/2}(t)X = 0$, $t > s$, $X(s) = I$ (see (4)), and by $V(t, s)$ ($0 \leq t \leq s \leq T$) the operator solution of the inverse Cauchy problem $\dot{X} - B^{1/2}(t)X = 0$, $t < s$, $X(s) = I$ (the substitution $t - s = \tau$ reduces this problem to the direct Cauchy problem). Let us pass to the boundary-value problem (1), (2). We shall seek its solution in the form

$$\begin{aligned} x(t) = & U(t, 0)b_1 + \int_0^t U(t, s)B^{-1/2}(s)y(s) ds + V(t, T)b_2 + \\ & + \int_t^T V(t, s)B^{-1/2}(s)y(s) ds. \end{aligned} \quad (3)$$

The elements b_1 and b_2 , as well as the abstract function $y(t)$, are to be determined. If $b_i \in D$, $i = 1, 2$, then $x(t)$ is strongly continuously differentiable on $[0, T]$. Differentiating (3) with respect to t , we obtain an expression for $\dot{x}(t)$. Putting in this expression and in (3) $t = 0$ and $t = T$, and then substituting $x(0)$, $x(T)$, $\dot{x}(0)$, and $\dot{x}(T)$ into (2), we obtain a system of two linear equations which permits expressing b_1 and b_2 in terms of $y(t)$:

$$\sum_{j=1}^2 A_{ij}b_j = C_i + \int_0^T H_i(s)B^{-1/2}(s)y(s) ds, \quad i = 1, 2. \quad (4)$$

Here A_{ij} , $H_i(s)$, and C_i are known linear operators, operator-functions, and elements of E , respectively.

Let now the following condition be satisfied.

- IV. For any strongly continuous functions $y(t)$ on $[0, T]$ and for any C_i , $i = 1, 2$, the system (4) has a solution (b_1, b_2) , with $b_i \in D$, $i = 1, 2$, if $C_i \in D$, $i = 1, 2$.

We note now that equality (3) can be twice continuously differentiated with respect to t on $(0, T)$, if $b_i \in D$, $i = 1, 2$, and $y(t)$ satisfies the weakened Hölder condition: for $0 < t \leq s < T$,

$$\|y(t) - y(s)\| \leq C_1 \frac{|t - s|^\nu}{t^\nu(T - s)^\nu}. \quad (5)$$

Assuming these conditions to be fulfilled and differentiating the expression for $\dot{x}(t)$ once more with respect to t , and taking into account (1), as well as the expressions for b_1 and b_2 , we obtain integral equations for determining $y(t)$:

$$y(t) = -\frac{1}{2}h(t) + \frac{1}{2}\dot{B}^{1/2}(t) \left\{ \int_0^T R(t, s)B^{-1/2}(s)y(s) ds + Y_1(t)a_1 + Y_2(t)a_2 \right\}, \quad (6)$$

where $R(t, s)$, $Y_1(t)$, and $Y_2(t)$ are known operator-functions. Finally, we require that the following condition be satisfied:

- V. The integral equation (6), for $a_i \in D$, $i = 1, 2$, has a solution $y(t)$ that is strongly continuous on $[0, T]$ and satisfies the Hölder condition (5).

Thus proved is

Theorem 1. *Let conditions I–V be fulfilled and let $a_i \in D$, $i = 1, 2$; then there exists a solution of problem (1), (2). It can be written in the form*

$$x(t) = \int_0^T G(t, s)h(s) ds + X_1(t)a_1 + X_2(t)a_2, \quad (7)$$

where $G(t, s)$ is the Green operator-function, and $X_i(t)$ is the operator solution of the homogeneous equation (1), such that $\gamma_i(X_j) = \delta_{ij}I$.

From Theorem 1 one can obtain various assertions on the existence of solutions of various boundary-value problems, and in considering con-

of concrete problems the conditions of Theorem 1 can be weakened. In particular, the following is true.

Theorem 2. *Suppose conditions I–III and the following conditions are satisfied:*

$$1) \quad Ne^{-\alpha T} < 1, \quad \frac{kN}{\alpha} \left[1 + \frac{(N-1)e^{-\alpha T/2}}{1 - Ne^{-\alpha T}} \right] < 1,$$

where $\alpha = \beta - 2kN$.

2) In the case of problem A_1 , suppose $a_i \in D$, $i = 1, 2$.

3) In the case of problem A_3 , suppose $B(0) = B(T)$.

Then the problems A_i , $i = 1, 2, 3$, are solvable.

For the proof we note that the first of the inequalities in condition 1) ensures the fulfillment of condition IV, and the second that of condition V (the applicability of the contraction mapping principle to equation (6)).

Remark 1. Condition 2) of Theorem 2 can be weakened by requiring that $a_1 \in D(B^{\varepsilon/2}(0))$, $a_2 \in D(B^{\varepsilon/2}(T))$ (see (1)), where $0 < \varepsilon < 1$; in this case the second of conditions 1) is slightly complicated.

Remark 2. If condition I is replaced by the stronger requirement that for all $t \in [0, T]$ the operator $-B(t)$ be the infinitesimal generator of a strongly continuous semigroup $\exp\{-\xi B(t)\}$, so that $\|\exp\{-\xi B(t)\}\| \leq Ne^{-\beta\xi}$, then $\|\exp\{-\xi B^{1/2}(t)\}\| \leq Ne^{-\beta\xi}$.

Remark 3. If $N = 1$, then condition 1) of Theorem 2 becomes the requirement that $K < 1/3\beta$.

The simplest case is when $B(t) \equiv B$ does not depend on t (cf. (5)). Repeating our reasoning, we arrive at system (4), where now all A_{ij} commute, and therefore one can use an analogue of Cramer's rule. We compute the determinant of the system Δ :

$$\begin{aligned} \Delta = \det(A_{ij}) = & \Gamma_{12}[I - S(2T)] + \\ & + B^{1/2}\{[\Gamma_{14} + \Gamma_{23}][I + S(2T)] + 2[\Gamma_{13} + \Gamma_{24}]S(T)\} - \Gamma_{13}B[I - S(2T)], \end{aligned}$$

where $S(t) = \exp\{-tB^{1/2}\}$, $\Gamma_{ij} = \begin{vmatrix} \gamma_{i1} & \gamma_{j1} \\ \gamma_{i2} & \gamma_{j2} \end{vmatrix}$. The other determinants are computed similarly.

Theorem 3. Suppose $B(t) \equiv B$; Δ^{-1} exists and is bounded; conditions I and III are satisfied, as well as one of the conditions: 1) $\Gamma_{34} \neq 0$; 2) $\Gamma_{34} = 0$, $\Gamma_{14} + \Gamma_{23} \neq 0$; 3) $\gamma_{i3} = \gamma_{i4} = 0$, $i = 1, 2$, $\Gamma_{12} \neq 0$. Then problem (1), (2) is solvable. Its solution is given by formula (3), where $y(t) = -1/2h(t)$, (b_1, b_2) is the solution of system (4).

Theorem 4. Suppose $B(t) \equiv B$, $Ne^{-\beta T} < 1$ and conditions I and III are satisfied; then the problems A_i , $i = 1, 2, 3$, are solvable.

The results obtained partially carry over to the equation

$$\ddot{x} + A(t)\dot{x} - B(t)x = h(t), \quad (8)$$

where $A(t)$ is a closed linear operator in E with domain dense in E , not depending on t . Suppose that, for every $t \in [0, T]$, $-A(t)$ is the infinitesimal generator of a strongly continuous group $\exp\{-\xi A(t)\}$. Introduce $\theta(t)$, the operator solution of the Cauchy problem: $\dot{\theta}(t) + {}^{1/2}A(t)\theta = 0$, $\theta(0) = I$. After the substitution $x = \theta(t)z$, equation (8) is reduced to an equation of the form (1) with operator

$$\widetilde{B}(t) = \theta^{-1}(t)\{ {}^{1/2}\dot{A}(t) + {}^{1/4}A^2(t) + B(t) \}\theta(t).$$

In this way we have found the solvability conditions for problem A_1 for (8).

We turn to questions of uniqueness of the solution of problems (1), (2). Consider in E^* the problem adjoint to problem (1), (2):

$$\ddot{\psi} - B^*(t)\psi = \varphi(t), \quad \gamma_i^*(\psi) = 0, \quad i = 1, 2. \quad (9)$$

Theorem 5. *Suppose problem (9) is solvable for arbitrary right-hand sides $\varphi(t)$ satisfying a Hölder condition of type III; then the solution of problem (1), (2) is unique.*

We note that from this follows the uniqueness of the solution of problem (1), (2), if E is a Hilbert space and the operator $B(t)$ and the boundary conditions

(2) are self-adjoint. In the general case, using the solvability conditions obtained above, it is not difficult to establish the solvability of the adjoint problem. In this way the following is proved.

Theorem 6. *Suppose that the following conditions are satisfied:*

I. Condition I is satisfied and, in addition, $D(B^*(t))$ is dense in E^* for each $t \in [0, T]$ (the latter condition is satisfied if E is reflexive).*

II. For $t, \tau \in [0, T]$ the operator $\overline{B^{-1/2}(\tau)B^{1/2}(t)}$ is continuously differentiable with respect to t in the uniform operator topology, and $B^{-1/2}(\tau)\dot{B}^{1/2}(t)$ satisfies the Hölder condition (see II).*

Then from the solvability of problem (1), (2) with self-adjoint boundary conditions there follows the uniqueness of its solution.

Let us note that the boundary conditions of the problems A_i , $i = 1, 2, 3$, are self-adjoint. In the case of problem A_1 , the uniqueness theorem is valid under weaker restrictions and is a consequence of one-sided estimates and the maximum principle.

Theorem 7. Consider the eigenvalue problem $\ddot{x} - B(t)x = \lambda x$ with homogeneous boundary conditions of one of the problems A_i , $i = 1, 2, 3$. Suppose that the hypotheses of Theorem 2, condition I and Π^* of Theorem 6 are satisfied, and suppose that $B^{-1}(t)$ is completely continuous for each $t \in [0, T]$. Then there exists a finite or countable set of eigenvalues; all of them are of finite multiplicity and isolated. If the set of eigenvalues is countable, then $|\lambda_n| \rightarrow +\infty$ as $n \rightarrow +\infty$.

The results obtained have applications to systems of ordinary differential equations, to integro-differential equations, and to elliptic equations.

Example. $E = L_p(\Omega)$, where Ω is a simply connected bounded domain in R^n with sufficiently smooth boundary Γ ,

$$B(t)x \equiv \sum_{i,j=1}^n a_{ij}(t, \xi) \frac{\partial^2 x}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^n a_i(t, \xi) \frac{\partial x}{\partial \xi_i} + a(t, \xi)x;$$

$t \in [0, T]$; $\xi = (\xi_1, \xi_n)$; a_{ij}, a_i

and a are sufficiently smooth and

$$\sum_{i,j=1}^n a_{ij}(t, \xi) q_i q_j \geq k \sum_{i=1}^n q_i^2, \quad k = \text{const} > 0.$$

The domain of definition of $B(t)$ consists of the functions $\dot{W}_p^{(2)}(\Omega)$. In the cylinder $\Omega \times (0, T)$ consider the elliptic equation

$$\frac{\partial^2 x}{\partial t^2} + \sum_{i,j=1}^n a_{ij}(t, \xi) \frac{\partial^2 x}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^n a_i(t, \xi) \frac{\partial x}{\partial \xi_i} + a(t, \xi)x = h(t, \xi)$$

with boundary conditions $x|_{\Gamma \times [0, T]} = 0$ and $\gamma_i(x) = d_i(\xi)$, $i = 1, 2$ (see (2)). By a solution of this problem we shall mean a solution of problem (1), (2). It is not difficult to rephrase the results obtained for this case.

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Moscow Institute of Physics and Technology

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