

TRANSFORMATION OPERATORS AND THE SCATTERING FUNCTION IN THE CASE OF A HIGHLY SINGULAR POTENTIAL

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Abstract

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MATHEMATICS

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TRANSFORMATION OPERATORS AND THE SCATTERING FUNCTION IN THE CASE OF A HIGHLY SINGULAR POTENTIAL

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1. We consider the radial scattering equation

$$y'' + \{k^2 - V(x)\}y = 0, \quad 0 < x < \infty, \quad (1)$$

with a repulsive potential of the form $V(x) = W(x) + U(x)$, where $W(x) \rightarrow +\infty$ sufficiently rapidly as $x \rightarrow 0$, and $U(x)$ is a relatively small perturbation. It is also assumed that for any $a > 0$

$$\int_a^\infty x|V(x)| dx < \infty. \quad (2)$$

Denote by $f(x, k)$ the solution of equation (1) determined by the condition

$$\lim_{x \rightarrow \infty} f(x, k)e^{-ikx} = 1. \quad (3)$$

It is known that, in solving the inverse problem of scattering theory and in a number of other questions, the application of transformation operators of the form

$$f(x, k) = e^{ikx} + \int_x^\infty K(x, t)e^{ikt} dt, \quad (4)$$

which were introduced by B. Ya. Levin ⁽¹⁾ and studied in detail in the works of Z. S. Agranovich and V. A. Marchenko ^(2,3), has proved very fruitful. In this setting, the singularity of the potential as $x \rightarrow 0$ was in any case allowed to be no higher than Cx^{-2} . We shall assume that the potential (more precisely, $W(x)$) has as $x \rightarrow 0$ a singularity of higher order,* than x^{-2} , that $W(x)$ is three times continuously differentiable for $x > 0$, and that as $x \rightarrow 0$

$$W'(x) = O(W^{3/2-\delta}(x)), \quad 0 < \delta < \frac{1}{2}; \quad (5)$$

$$W''(x)/W'(x) = O(W'(x)/W(x)), \quad W'''/W'' = O(W''/W'). \quad (6)$$

Let, moreover, $W(x)$, together with its three derivatives, be monotone for $0 < x < 1$, let $W(x) \equiv 0$ for $x > 1$, and let the function $U(x)$ be measurable and

$$\int_0^{1/2} |U(x)| \cdot |W'(x)|^{-1/3} dx < \infty. \quad (7)$$

Conditions (5), (6) are satisfied, in particular, for functions of the form Cx^{-p} for any $p > 2$, $C > 0$ (with its own δ for each p), and also for $e^{1/x}$, $\exp\{e^{1/x}\}$, etc.

* This follows from the remaining conditions.

We denote

$$\gamma(x) = 2^{-1/2} W^{-1/4}(x) \exp \left\{ \int_x^1 W^{1/2}(t) dt \right\}, \quad (8)$$

$$\gamma_0(x) = 2^{-1/2} W^{-1/4}(x) \exp \left\{ - \int_x^1 W^{1/2}(t) dt \right\}. \quad (9)$$

It is easy to see that $\gamma(x) \rightarrow +\infty$, $\gamma_0(x) \rightarrow 0$ as $x \rightarrow 0$.

Theorem 1. Let $K(x, t)$ be the kernel of the transformation operator (4) for equation (1) with potential $V(x) = W(x) + U(x)$, satisfying the conditions listed above. Then, as $x \rightarrow 0$, there exists the limit

$$\lim_{x \rightarrow 0} K(x, t)/\gamma(x) = K(t), \quad 0 < t < \infty, \quad (10)$$

where $K(t) \not\equiv 0$ is an infinitely differentiable function, bounded together with each of its derivatives $K^{(n)}(t)$ in the metrics $C[0, \infty)$ and $L^p[0, \infty)$ for any $p \geq 1$, and

$$K(0) = K^{(n)}(0) = 0, \quad n = 1, 2, \dots \quad (11)$$

Convergence to the limit (10) takes place in $C[0, \infty)$ and in $L^p[0, \infty)$ for $p \geq 1$.

The main stages of the proof are outlined below in Sec. 3.

2. Equation (1), under conditions close to ours, was studied in the works of Limić ⁽⁴⁾. In particular, he constructed a solution $\varphi(x, k)$, regular as $x \rightarrow 0$, for which

$$\lim_{x \rightarrow 0} \varphi(x, k)/\gamma_0(x) = 1, \quad (12)$$

defined the Jost function

$$f(k) = W\{f(x, k), \varphi(x, k)\} = f\varphi' - f'\varphi, \quad (13)$$

and also established the existence of the scattering function

$$S(k) = \lim_{x \rightarrow 0} f(x, -k)/f(x, k) = f(-k)/f(k). \quad (14)$$

These results of Limić are also valid in our case.

Applying similar arguments, one can also establish the following theorem.

Theorem 2. For any solution $y(x, k)$ of equation (1) with a potential of the kind considered, satisfying condition (5) and the condition

$$\int_0^{1/2} |U(x)| W^{-1/2}(x) dx < \infty, \quad (15)$$

there exist equal finite limits

$$\lim_{x \rightarrow 0} y(x, k)/\gamma(x) = \lim_{x \rightarrow 0} y'(x, k)/\gamma'(x) = y(k), \quad (16)$$

where $y(k) = 0$ only in the case when $y(x, k) = C\varphi(x, k)$, and then

$$\lim_{x \rightarrow 0} y'(x, k)/\gamma'_0(x) = \lim_{x \rightarrow 0} y(x, k)/\gamma_0(x). \quad (17)$$

We note that condition (15) is weaker than (7), in view of (5).

Corollary 1. For any solution $y(x, k)$

$$\lim_{x \rightarrow 0} W^{-1/2}(x) y'(x, k)/y(x, k) = \mp 1, \quad (18)$$

where the sign $+$ occurs only if $y(x, k) = C\varphi(x, k)$.

Corollary 2. The Jost function (13) is equal to

$$f(k) = \lim_{x \rightarrow 0} f(x, k)/\gamma(x). \quad (19)$$

3. Denote by $X(k)$ the root of the equation

$$W(X) = k^2, \quad (20)$$

$$\alpha(k) = -k + \int_{X(k)}^1 \{k^2 - W(t)\}^{1/2} dt, \quad k > 0. \quad (21)$$

Theorem 3. Under the conditions of Theorem 1, as $k \rightarrow +\infty$,

$$f(k) \sim \gamma^{-1}(X) \exp \left\{ -i \left[\frac{\pi}{4} + \alpha(k) \right] - k^2 \int_0^X \frac{dt}{\sqrt{W(t) - k^2} + \sqrt{W(t)}} \right\}, \quad (22)$$

and for some $C > 0$, $C_1 > 0$,

$$|f(k)| \leq C_1 \exp\{-C|k|^{2\delta}\}, \quad -\infty < k < \infty. \quad (23)$$

The **proof** is based on the study of the integral equation for $f(x, k)$ by means of Langer's method ⁽⁵⁾, which is generalized to the case of potentials of arbitrary rate of growth as $x \rightarrow 0$.

Let

$$\Phi_1(x, k) = f(x, k)/\gamma(x), \quad \Phi_2(x, k) = \Phi_1(x, k) - e^{ikx}/\gamma(x),$$

$$\Phi_3(x, k) = \Phi_2(x, k) - \frac{1}{\gamma(x)} \int_x^\infty \frac{\sin k(t-x)}{k} V(t) e^{ikt} dt.$$

Lemma 1. The convergence $\lim_{x \rightarrow 0} \Phi_j(x, k) = f(k)$ takes place in the metric $C(-\infty, \infty)$ for all $j = 1, 2, 3$, and also in $L^p(-\infty, \infty)$ for $\Phi_2(x, k)$ when $p > 1$ and for $\Phi_3(x, k)$ when $p > 1/2$.

Lemma 2. For any $n \geq 1$,

$$\sup_{-\infty < k < \infty} \left| k^n \frac{d}{dk} f(k) \right| < \infty. \quad (24)$$

Theorem 3, Lemmas 1, 2, and the theorem on convergent sequences of Hermitian-positive functions ⁽⁶⁾ lead to a complete proof of Theorem 1, if the following is taken into account.

Remark. If $V(x) \geq 0$, then $f(k)$ is Hermitian-positive. If $V_2(x) \geq |V_1(x)|$, then $K_2(x, t) \geq |K_1(x, t)|$, where K_1 and K_2 are the kernels of the corresponding transformation operators.

4. **Theorem 4.** Under the conditions of item 1, the Jost function $f(k)$ admits the integral representation

$$f(k) = \int_0^\infty K(t) e^{ikt} dt, \quad (25)$$

where $K(t)$ is defined by (10) and possesses all the properties indicated in Theorem 1.

The following theorem follows from Theorem 3 and makes it possible, from the asymptotics of the scattering function $S(k)$, to find the asymptotics of the potential as $x \rightarrow 0$.

Theorem 5. Under the conditions of item 1, when $k \rightarrow +\infty$,

$$S(k) = \exp\{i[2\alpha(k) + \pi/2 + o(1)]\}, \quad (26)$$

where $\alpha(k)$ is defined by (21) and, as is known, $S(-k) = S^{-1}(k)$. If, moreover, there exists, for some $p > 2$,

$$\lim_{x \rightarrow 0} W(x) x^p = C \neq 0,$$

then, as $k \rightarrow +\infty$,

$$S(k) = i \exp\{-2ikX(k)[A_p + o(1)]\}, \quad (27)$$

where

$$A_p = 1 + \frac{1}{p} \int_0^1 (1 - \sqrt{1-t}) t^{-1-1/p} dt. \quad (28)$$

Formulas (27), (28) are also valid when, for every $\varepsilon > 0$, as $x \rightarrow 0$ $W(x)x^{p+\varepsilon} \rightarrow 0$, $W(x)x^{p-\varepsilon} \rightarrow \infty$ monotonically in some neighborhood of $x = 0$. Finally, if $[\ln W(x)]/|\ln x| \rightarrow \infty$ monotonically as $x \rightarrow 0$, then in (27) one should put $A_p = A_\infty = 1$.

For an analytic potential $V(z)$, admitting for $\operatorname{Re} z > 0$ the estimate $|V(z)| < C|z|^{-4}$, an asymptotic formula similar to (26) was obtained in ⁴. (There, however, the term $\pi/2$ was omitted.)

5. Let us consider the Cauchy problem for the equation

$$F''_{xx}(x, y) - V(x)F(x, y) = F''_{yy}(x, y) \quad (29)$$

with initial data $F(1, y)$, $F'_x(1, y)$, $-\infty < y < \infty$, which are generalized functions ⁷ over the basic space K of finite, infinitely differentiable functions. For each $x > 0$ the solution $F(x, y)$ is a generalized function over K .

Theorem 6. If $V(x)$ in (29) satisfies the conditions of item 1, then as $x \rightarrow 0$ $F(x, y)/\gamma(x) \rightarrow F(y)$ (in the sense of convergence of generalized functions), where $F(y)$ turns out to be an ordinary, infinitely differentiable function.

There exist initial data for which $F(y) \neq 0$, but if*

$$F(x, y) = (R(k), \varphi(x, k)e^{iky}), \quad R(k) \in Z', \quad \varphi(x, k)$$

is defined by (12), then $F(y) \equiv 0$.

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REFERENCES

1. B. Ya. Levin, DAN, 106, No. 2 (1956).
2. Z. S. Agranovich, Uchen. zap. Kharkov. gos. ped. inst., 21, matem. ser., iss. 2, 3 (1957).
3. S. Z. Agranovich, V. A. Marchenko, Inverse Problem of Scattering Theory, Kharkov, 1960.
4. N. Limić, Nuovo Cimento, 26, No. 3, 581 (1962); 28, No. 5, 1066 (1963).
5. E. C. Titchmarsh, Eigenfunction Expansions, 2, Moscow, 1961.
6. S. Bochner, Lectures on Fourier Integrals, Moscow, 1962.
7. I. M. Gel' fand, G. E. Shilov, Generalized Functions, vols. 1-3, Moscow, 1958.

* From (13) and (23) it follows easily that for any $x > 0$, $\varphi(x, k) \in Z$. For the definition of Z, Z' , see ⁷.

Note: Figure translations are in progress. See original paper for figures.

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