

ON THE RICCI CURVATURE OF THE BERGMAN METRIC INVARIANT UNDER BIHOLOMORPHIC MAPPINGS

B. A. Fuks

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.57854>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.55

MATHEMATICS

B. A. FUKS

ON THE RICCI CURVATURE OF THE BERGMAN METRIC INVARIANT UNDER BIHOLOMORPHIC MAPPINGS

(Presented by Academician M. A. Lavrent'ev, 12 VII 1965)

1. Let B be a bounded domain in the space C^n of complex variables $(z^1, \dots, z^n) = z$. Then $((^{1,2}); (^3)$, pp. 101-104, 346-350) the positive form

$$ds^2 = T_{k\bar{l}} dz^k \overline{dz^l}, \quad T_{k\bar{l}} = (\ln K)''_{z^k \bar{z}^l} \quad (1)$$

defines in the domain B the Bergman metric, invariant under biholomorphic mappings. Here $K(z, \bar{z}) = I_1^{-1}$ is the Bergman kernel function of the domain B ; I_1 is the minimal value of the integral

$$I = \int_B |f(\xi)|^2 d\omega \quad (2)$$

(where $d\omega$ is the volume element of the domain $B \subset C^n$, $\xi = (\xi^1, \dots, \xi^n)$) for functions $f(\xi) \in L^2(B)$ satisfying the condition $f(z) = 1$ at the point $z \in B$. The kernel function $K(z, \bar{z})$ is a real analytic function of the variables $x^k = \operatorname{Re} z^k$, $y^k = \operatorname{Im} z^k$, $k = 1, \dots, n$, in the domain B . It can also be defined by the equality $K(z, \bar{z}) = \sum_{\nu=1}^{\infty} |\varphi_{\nu}(z)|^2$, where $\{\varphi_{\nu}(z), \nu = 1, 2, \dots\}$ is any closed orthonormal system of holomorphic functions in $L^2(B)$.

The metric (1) is a special case of the Kähler metric

$$ds^2 = g_{k\bar{l}} dz^k \overline{dz^l}, \quad g_{k\bar{l}} = \Phi''_{z^k \bar{z}^l}, \quad (3)$$

defined in a domain $E \subset C^n$ by means of a function Φ , called the potential of the Kähler metric (3). Here Φ is an arbitrary plurisubharmonic function (which ensures the positivity of the form (3)). Suppose that in some domain $E \subset C^n$ a Kähler metric (3) is given by means of a real analytic potential Φ . The question arises whether there exists such a bounded domain $B \subset C^n$, with $B \cap E \neq \emptyset$,

that at the points $z \in B \cap E$ the metric (3) coincides with the metric (1) for B . The following theorem gives a necessary condition.

Theorem 1 (Fuks ^(3,4)). *If a Kähler metric is the Bergman metric for a bounded domain B , then its Riemannian curvature at all points $z \in B$ in any two-dimensional analytic direction*

$$R = |u|^{-4} R_{q\bar{l}p\bar{m}} \bar{u}^q u^l u^p \bar{u}^m < 2. \quad (4)$$

Here u^k is the vector defining this direction, $|u|$ is the length of this vector in the metric (3), and $R_{q\bar{l}p\bar{m}}$ is the curvature tensor of the metric (3).

Inequality (4) follows from the relation

$$R = 2 - K J^{-1} J_1^2, \quad (5)$$

which holds for the metric (1). Here $J_1 = (K|u|^2)^{-1}$ and J are the minimal values of the integral (2) in the class of functions $f(\zeta) \in L^2(B)$ satisfying, at the point $z \in B$, the conditions: for J_1 , $f(z) = 0$, $f'_{z^k} u^k = 1$; for J ,

$$f(z) = f'_{z^1} = \dots = f'_{z^n} = 0, \quad f''_{z^k z^l} u^k u^l = 1.$$

The assertion of Theorem 1 follows from the fact that $J_1, J > 0$.

This result was extended by Hua Lo-keng ⁽⁵⁾ to the case of a metric defined by means of the kernel function of an incomplete orthonormal system. Stark ⁽⁵⁾ obtained for the quantity R an expression different from (5), and showed that there are cases in which, starting from it, one can obtain a better estimate for R than from (5). Other generalizations of Theorem 1 were obtained by Hua Lo-keng ⁽⁷⁾, Matsuura ⁽⁸⁾, Mishiaki ⁽⁹⁾, and others.

2. Inequality (4) is difficult to use in studying the topological properties of the Kähler manifold into which the domain B is transformed as a result of introducing the metric (1). The reason is that the Riemann curvature is expressed through a form of the fourth degree, and finding its extremal values is difficult. Usually such investigations (see ⁽¹⁰⁾) are connected with considering, at points $z \in B$, the Ricci curvature

$$\rho = |u|^{-2} R_{p\bar{q}} u^p \bar{u}^q, \quad (6)$$

which is expressed by a quadratic form. Here $R_{p\bar{q}} = -(\ln D)''_{z^p \bar{z}^q}$ is the Ricci curvature tensor of the metric (3), $D = \det g_{k\bar{l}}$.

In the case of the metric (1), $D = K^{-(n+1)} \Delta$; $\Delta = \det K_{s\bar{t}}$, where $s, t = 0, 1, \dots, n$;

$$K_{s\bar{t}} = K''_{z^s \bar{z}^t} \quad \text{for } s, t > 0; \quad K_{s0} = K'_{z^s}, \quad K_{0\bar{t}} = K'_{\bar{z}^t}, \quad K_{00} = K.$$

There holds

Theorem 2. *If a Kähler metric is the Bergman metric for a bounded domain $B \subset C^n$, then at any point $z \in B$, in the direction of an arbitrary vector u^k , the Ricci curvature satisfies $\rho < n + 1$.*

The proof of this theorem is based on the relations

$$R_{p\bar{q}} = (n + 1)T_{p\bar{q}} - (\ln \Delta)''_{z^p \bar{z}^q}; \quad (7)$$

$$\rho = (n + 1) - (K\Delta|u|^2)^{-1}T^{p\bar{q}}S_{p\bar{q}}; \quad (8)$$

$$S_{p\bar{q}} = \begin{vmatrix} \Delta & K_{q\bar{l}}\bar{u}^l \\ \vdots & \vdots \\ K_{pk}u^k & \dots & K_{p\bar{n}k}u^k & K_{pk\bar{q}l}u^k\bar{u}^l \end{vmatrix}. \quad (9)$$

Here

$$K_{p\bar{k}s} = K'''_{z^p z^k \bar{z}^s}, \quad K_{s\bar{q}l} = K'''_{\bar{z}^s z^q \bar{z}^l}, \quad K_{pk\bar{q}l} = K^{IV}_{z^p z^k \bar{z}^q \bar{z}^l}.$$

Equality (8) is obtained as a result of computations from equalities (6) and (7).

Apply to the domain B such a linear biholomorphic mapping that the forms (1), (6), at an arbitrarily chosen fixed point $z \in B$, will have canonical form. We shall keep the old notation for the domain obtained as a result of this mapping, for the point corresponding to the point z , and for the quantities connected with the Bergman metric. In this case the quantity ρ , thanks to the invariance of the metric (1) under biholomorphic mappings, will have its former value. Since now $T^{p\bar{q}}(z, \bar{z}) = 0$ for $p \neq q$, in our case

$$\rho = (n + 1) - (K\Delta|u|^2)^{-1}T^{p\bar{p}}S_{p\bar{p}}. \quad (10)$$

A proper calculation shows that $S_{p\bar{p}} = \Delta(J^{(p)})^{-1}$, where $J^{(p)}$ is the minimal value of the integral (3) for the class of functions $f(\zeta) \in L^2(B)$,

satisfying the conditions $f(z) = f'_{z_1} = \dots = f'_{z_n} = 0$, $f''_{z_s z_p} u^s = 1$. This calculation is based on a single general formula of S. Bergman (see ⁽³⁾, formula (1.58)). The quantity $J^{(p)} > 0$. Hence, from the fact that $T^{p\bar{p}} > 0$ (where $p = 1, \dots, n$), and from formula (10), our assertion follows.

Let us note that Theorem 2 can also be proved in another way. To avoid making the exposition excessively complicated, we shall henceforth restrict ourselves to

the case of two complex variables. Then, starting from equality (7), one can show that for $n = 2$ (in an arbitrary coordinate system)

$$\rho = 3 - J_1 I_1^2 I_2^{-1} [(I_2 J_2)^{-1} + (I_3 J_3)^{-1}]. \quad (11)$$

Here I_1 and J_1 are the quantities defined above; I_2, I_3, J_2, J_3 are the minimum values of the integral (2) in the class of functions $f(\zeta) \in L^2(B)$ satisfying at the point $z \in B$ the following conditions: for I_2 , $f(z) = 0$, $f'_{z_1} = 1$; for I_3 , $f(z) = f'_{z_1} = 0$, $f'_{z_2} = 1$; for J_2 , $f(z) = f'_{z_1} = f'_{z_2} = 0$, $f''_{(z_1)^2} u^1 + f''_{z_1 z_2} u^2 = 1$; for J_3 , $f(z) = f'_{z_1} = f'_{z_2} = 0$, $(-f''_{(z_1)^2} T'_{21} + f''_{z_1 z_2} T'_{11}) u^1 + (-f''_{z_1 z_2} T'_{21} + f''_{(z_2)^2} T'_{11}) u^2 = T'_{11}$. From formula (11), since $I_s, J_s > 0$ ($s = 1, 2, 3$), it follows that $\rho < 3$; for $n = 2$ this is precisely the assertion of the theorem.

Corollary 1. *If the Kähler metric is the Bergman metric for a bounded domain $B \subset C^n$, then at every point $z \in B$ its scalar curvature is $< n(n+1)$.*

This estimate follows from the fact that the scalar curvature is equal to the sum of the Ricci curvatures in the directions of n mutually orthogonal vectors.

Corollary 2. *In a bounded domain $B \subset C^n$ the Kähler metric*

$$ds_1^2 = \tau_{k\bar{l}} dz^k d\bar{z}^l, \quad (1')$$

where $\tau_{k\bar{l}} = (\ln \Delta)''_{z_k \bar{z}_l}$, $\Delta = K^{n+1} D$, and D is the discriminant of the form (1), is positive and invariant under biholomorphic mappings.

The positivity of the form (1') follows immediately from Theorem 1; the invariance of the form (1') follows from the fact that K and D are tensor densities under biholomorphic mappings.

3. We shall consider the behavior of the Ricci curvature of the Bergman metric at a point $z \in B$ as the point z approaches the boundary of the domain $B \subset C^2$. We restrict ourselves to the space C^2 , since the method we use was developed ((²), pp. 6-27) only for C^2 . This method is based on the following fact. Let bounded domains $g, B, G \subset C^n$ be such that $g \subset B \subset G$, and let $z \in g$; let I_g, I_B, I_G be the minimum values of integrals of the form (2) for these domains under the same additional conditions at the point z . Then (see (³), p. 100, formula (1.63))

$$I_g \leq I_B \leq I_G. \quad (12)$$

We shall take as g and G domains obtained from a hypersphere by certain biholomorphic mappings. Let $z \in g$. Then we apply to the domains g, B, G such a linear biholomorphic mapping that the forms (1) for the domains B and G at the point z simultaneously pass into forms having canonical form. We retain, for the domains obtained as a result of this mapping, for the point corresponding

to z , and for the quantities related to the Bergman metric in these domains, the previous notation. Now we have $T_{12}^{(B)}(z, \bar{z}) = T_{12}^{(G)}(z, \bar{z}) = 0$, and the conditions determining the minimum value J_3 of the integral (2) for the domains B and G become the same. The conditions determining the other minimum values of the integral (2) at the point z , appearing in formula (11), in general do not depend on the domain. Therefore, for the domains obtained as a result of our mapping, by virtue of (11) and (12),

$$3 - \rho \geq J_1^{(g)}(I_1^{(g)})^2(I_2^{(G)})^{-1} [(I_2^{(G)} J_2^{(G)})^{-1} + (I_3^{(G)} J_3^{(G)})^{-1}] = Q(3 - \rho_G) = 4Q;$$

$$Q = J_1^{(g)}(I_1^{(g)})^2(J_1^{(G)})^{-1}(I_1^{(G)})^{-2}. \quad (13)$$

The penultimate equality follows from the fact that the Bergman metric in the hypersphere (and in domains biholomorphically equivalent to the hypersphere) has constant Riemannian curvature $R = -2/3^*$; in this case (see, for example, ⁽¹⁰⁾, p. 103) the Riemann and Ricci curvatures are related by the formula $\rho = 1/2(n+1)R$, and therefore $\rho_G = \rho_g = -1$ (in our case $n = 2$).

ρ is an invariant of biholomorphic mappings. Under a biholomorphic mapping the quantity $J_1 I_1$ is multiplied by $|j|^{-6}$, where j is its Jacobian. This follows from the transformation law for the kernel function of a domain under biholomorphic mappings. Therefore Q is an invariant of biholomorphic mappings, and inequality (13) is preserved under them for the original domains g, B, G . Thus,

$$4Q \leq 3 - \rho \leq 4Q^{-1}. \quad (14)$$

The right-hand side of this inequality can be obtained in the same way as the left-hand side. For this purpose one must apply to the domains g, B, G a linear biholomorphic mapping that brings the forms (1) for the domains g and B at the point z simultaneously into canonical form, and then repeat the arguments used to obtain the left-hand side of the inequality.

Theorem 3. *If $B \subset C^2$ is a bounded domain, and: 1) the boundary ∂B is a hypersurface of class $\mathfrak{C}^2$; 2) at the point $\zeta \in \partial B$ the hypersurface ∂B is strictly convex in the sense of Levi (i.e. the Levi determinant $\mathcal{L}(\partial B)|_{z=\zeta} > 0$) and one of the analytic surfaces passing through the point ζ lies entirely outside the domain B ; 3) the point $z \in B$ approaches the point ζ , remaining inside the cone formed by rays issuing from the point ζ and making with the inner normal to the hypersurface ∂B an angle not greater than α (where α is some number satisfying $0 < \alpha < \pi/2$), then $\lim_{z \rightarrow \zeta} \rho = -1$.*

To prove this theorem we take in inequality (14) $g = \mathfrak{I}$, $G = \mathfrak{A}$, where $\mathfrak{I}$ and $\mathfrak{A}$ are the so-called inner and outer comparison domains corresponding to the point ζ ; they are obtained from the hypersphere by biholomorphic mappings,

$\mathfrak{J} \subset B \subset \mathfrak{A}$ (therefore inequality (14) holds for them). The point ζ belongs to the boundary of all three domains, and when $z \rightarrow \zeta$ in the manner indicated in the theorem,

$$\lim_{z \rightarrow \zeta} J_1^{(\mathfrak{J})} = \lim_{z \rightarrow \zeta} I_1^{(\mathfrak{J})} = \lim_{z \rightarrow \zeta} J_1^{(\mathfrak{A})} = \lim_{z \rightarrow \zeta} I_1^{(\mathfrak{A})} = 1.$$

Our assertion follows from this.

Remark. The relation $\lim_{z \rightarrow \zeta} \rho = -1$ also holds in the case when $\mathcal{L}(\partial B)|_{z=\zeta} = 0$. However, in this case one must consider a way of approaching the point z to the point ζ different from that indicated in theorem (3) (this method of approach is described in ⁽²⁾, p. 20 and following).

Moscow Institute of Electronic
Machine Building

Received
5 VI 1965

CITED LITERATURE

- ¹ S. Bergman, *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications*, N. Y., 1941.
- ² S. Bergman, *Sur la fonction-noyau d' un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Paris, 1948.
- ³ B. A. Fuks, *Special Chapters of the Theory of Analytic Functions of Several Complex Variables*, Moscow, 1963.
- ⁴ B. A. Fuks, *Mat. sborn.*, 2 (44), 567 (1937).
- ⁵ Hua Loo-Keng, *On the Riemannian Curvature in the Space of Several Complex Variables*, Schr. Forschungsinst. Math., 1957, S. 245-263.
- ⁶ J. M. Stark, *Pacific J. Math.*, 10, 1021 (1960).
- ⁷ Hua Loo-Keng, *Acta Sci. Sinica*, 4, No. 1, (1955).
- ⁸ S. Matsuura, *Sci. Rep. Tokyo Kyoiku Daigaku*, A. 7, 5, March, 231 (1963).
- ⁹ Y. Michiwaki, *Res. Rep. Nagaoka Techn. Coll.*, 1, No. 1, 11 (1962).
- ¹⁰ K. Yano, S. Bochner, *Curvature and Betti Numbers*, IL, 1957.

* Let us note that in paper ⁽⁴⁾ this formula erroneously gives -1 instead of $-2/3$. This error was then repeated in the reviews ^(2, 3).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.