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Abstract

Full Text

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MATHEMATICS

N. S. SINYUKOV

ON THE THEORY OF GEODESIC MAPPINGS OF RIEMANNIAN SPACES

(Presented by Academician A. N. Kolmogorov on 13 XI 1965)

The paper considers and in principle solves the problem of determining all Riemannian spaces admitting a nontrivial geodesic mapping onto a given V_n . In addition, an estimate is given for the degree of arbitrariness in the solution of this problem, and necessary and sufficient conditions of algebraic character are obtained, in invariant form, for V_n 's admitting a nontrivial geodesic mapping. The investigation is carried out in the class of analytic functions, locally, without restriction on the signature of the spaces.

1. A Riemannian space V_n with metric tensor g_{ij} ($i, j = 1, 2, \dots, n$) admits a nontrivial geodesic mapping if and only if there exists a solution $\check{g}_{ij}$ ($= \check{g}_{ji}$) of the system of equations

$$\check{g}_{ij,k} = \lambda_i g_{kj} + \lambda_j g_{ki}, \quad (A_1)$$

where $\lambda_i \neq 0$ is a gradient vector, and the comma denotes the sign of covariant differentiation in V_n . In this case the metric tensor $\bar{g}_{ij}$ of the corresponding space $\bar{V}_n$ is determined from the relations

$$\bar{g}_{ij} = e^{2\psi} \check{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \quad \psi_{,i} = -\lambda_\alpha \check{g}^{\alpha\beta} g_{\beta i}, \quad (1)$$

where $\check{g}^{ij}$ are the elements of the inverse matrix for $\|\check{g}_{ij}\|$ (1). Nonvanishing of the determinant of the solution $\check{g}_{ij}$ of equations (A_1) can always be achieved by adding to it a term of the form $c g_{ij}$ ($c = \text{const}$).

From the integrability conditions of system (A_1) we find

$$\check{g}_{\alpha j} R^\alpha_{ikl} + \check{g}_{\alpha k} R^\alpha_{ilj} + \check{g}_{\alpha l} R^\alpha_{ijk} = 0, \quad (2)$$

$$\check{g}_{\alpha i} R^\alpha_{j} - \check{g}_{\alpha j} R^\alpha_{i} = 0, \quad (3)$$

$$n\lambda_{i,j} = \mu g_{ij} + \check{g}_{\alpha i} R^\alpha_j - \check{g}_{\alpha\beta} R^\alpha_{ji}{}^\beta, \quad (A_2)$$

as a consequence of which they reduce to the form

$$\check{g}_{\alpha_3} T^{\alpha_3}_{ij kl} = 0, \quad (B_1)$$

$$T^{\alpha\beta}_{ij kl} = n \left(\delta_i^\alpha R^\beta_{jkl} - \delta_k^\alpha R^\beta_{l(ij)} \right) - g_{jk} \left(\delta_i^\alpha R^\beta_l - R^\alpha_{il}{}^\beta \right) + g_{il} \left(\delta_k^\alpha R^\beta_j - R^\alpha_{kj}{}^\beta \right).$$

In turn, from the integrability conditions of equations A_2 it follows that

$$(n-1)\mu_{,k} = 2(n+1)\lambda_\alpha R^\alpha_k - \check{g}_{\alpha\beta} (R^{\alpha\beta}_{,k} - 2R^{\alpha}_{k,}{}^\beta), \quad (A_3)$$

and they are represented in the form

$$\begin{aligned} (n+3)\lambda_\alpha R^\alpha_{ilk} &= \frac{1}{n-1} g_{il} \left[(n+3)\lambda_\alpha R^\alpha_k - \check{g}_{\alpha_3} (R^{\alpha_3}_{,k} - 2R^{\alpha}_{k,}{}^\beta) \right] \\ &\quad - \frac{1}{n-1} g_{ik} \left[(n+3)\lambda_\alpha R^\alpha_l - \check{g}_{\alpha_3} (R^{\alpha_3}_{,l} - 2R^{\alpha}_{l,}{}^\beta) \right] \\ &\quad + \check{g}_{\alpha i} R^\alpha_{[l,k]} + \check{g}_{\alpha_3} R_{lki}{}^{\beta,\alpha}. \end{aligned} \quad (B_2)$$

Finally, the integrability conditions of the equations A_3 (taking into account A_1 and A_2) give the relations

$$\frac{n+1}{n} \check{g}_{\alpha\beta} (R^{\alpha}_{k\gamma}{}^\beta R^\gamma_l - R^{\alpha}_{l\gamma}{}^\beta R^\gamma_k) + (n+3)\lambda_\alpha R^\alpha_{[k,l]} - \check{g}_{\alpha_3} \left(\frac{1}{2} R^{\alpha\beta}_{,[kl]} - R^{\alpha}_{k,}{}^\beta + R^{\alpha}_{l,}{}^\beta \right). \quad (B_3)$$

Thus, we obtain a system of differential equations A_{123} of the first order with respect to g_{ij} , λ_i , μ , solved with respect to their derivatives, with coefficients from V_n and integrability conditions B_{123} , and hence also the theorem:

Theorem 1. *In order that V_n admit a nontrivial geodesic mapping, it is necessary and sufficient that the system of equations A have a nontrivial solution $\check{g}_{ij} \neq cg_{ij}$, $\lambda_i \neq 0$.*

Since the general solution of the system A is determined from it (in the form of a Taylor series) uniquely up to the initial conditions $\overset{0}{g}_{ij}$, $\overset{0}{\lambda}_i$, $\overset{0}{\mu}$, which can be chosen arbitrarily only in the case of its complete integrability, it follows from B that

Theorem 2. *The general space $\bar{V}_n$ admitting a nontrivial geodesic mapping onto a given V_n is determined by this space uniquely up to $r \leq (n+1)(n+2)/2$ arbitrary parameters. This maximum is attained if and only if V_n is a space of constant curvature.*

The relations B , the first B^1 , the second B^2 , and so on, their continuations, are linear algebraic equations with respect to $\check{g}_{ij}$, λ_i , and μ , with coefficients from V_n . Since the solution of the system A corresponding to initial values satisfying the indicated relations satisfies them identically, we obtain the theorem:

Theorem 3. *In order that V_n admit a nontrivial geodesic mapping, it is necessary and sufficient that the system of equations B , B^1 , B^2 , ... have a nontrivial solution with respect to $\check{g}_{ij}$ and λ_i .*

Analysis of the result of covariant differentiation of the conditions B_1 shows that when $\check{g}_{\alpha\beta} T_{ij\,kl,m}^{\alpha\beta} = 0$ by virtue of B_1 , $\lambda_i = 0$. This occurs for V_n satisfying the relations

$$T_{ij\,kl,m}^{(\alpha 3)} = Q_{ijklm}^{pqrs} T_{pq\,rs}^{(\alpha 3)}. \quad (4)$$

Consequently, we have the theorem:

Theorem 4. *Riemannian spaces satisfying conditions (4) do not admit a nontrivial geodesic mapping.*

2. Consider the relations (2), writing them in the form

$$\check{g}_{\alpha 3} \tilde{T}_{ij\,kl}^{\alpha 3} = 0,$$

$$\tilde{T}_{ij\,kl}^{\alpha 3} = \delta_j^{(\alpha} R^{3)}_{ikl} + \delta_k^{(\alpha} R^{\beta)}_{ilj} + \delta_l^{(\alpha} R^{3)}_{ijk}. \quad (5)$$

In the matrix $\|\tilde{T}_{ij\,kl}^{\alpha\beta}\|$ we shall take combinations of upper indices as the column numbers, and the lower indices as the row numbers. If $\alpha_1, \alpha_2, \dots, \alpha_n$ are distinct numbers from 1 to n , then formula (5) shows that the determinant of the minor of order $(n-2)$ of the matrix corresponding to the column numbers $(\alpha_3\alpha_3), (\alpha_3\alpha_4), \dots, (\alpha_3\alpha_n)$ and to the row numbers $(\alpha_1\alpha_3, \alpha_1\alpha_2), (\alpha_1\alpha_4, \alpha_1\alpha_2), \dots, (\alpha_1\alpha_n, \alpha_1\alpha_2)$ is

$$\Delta = 2 \left(R^{\alpha_3}_{\alpha_1\alpha_1\alpha_2} \right)^{n-2}. \quad (6)$$

For the determinant of the minor of order $(n-2)$ of the same matrix corresponding to the column numbers $(\alpha_1\alpha_4), (\alpha_4\alpha_4), \dots, (\alpha_4\alpha_n)$ and to the row numbers $(\alpha_1\alpha_1, \alpha_2\alpha_3), (\alpha_1\alpha_4, \alpha_2\alpha_3), \dots, (\alpha_1\alpha_n, \alpha_2\alpha_3)$, we obtain the formula

$$D_1 = 2 \left(R^{\alpha_4}_{\alpha_1\alpha_2\alpha_3} \right)^{n-2}. \quad (7)$$

Hence it follows

Lemma 1. If the rank of $\|\tilde{T}_{ijkl}^{\alpha\beta}\| \rho < n - 2$, then V_n is a space of constant curvature.

A joint consideration of relations B^1 and B^2 leads to the lemma:

Lemma 2. For V_n distinct from spaces of constant curvature, λ_i and μ are determined uniquely in terms of the objects V_n , the components of the tensor g_{ij} , and one of the components of the vector λ_i .

Lemmas 1 and 2 give the theorem:

Theorem 5. The totality of all Riemannian spaces $\bar{V}_n$ admitting a nontrivial geodesic mapping onto V_n , distinct from a space of constant curvature, depends on

$r \leq n(n+1)/2 - (n-4)$ essential parameters.

In a similar way, from B^1 , B^2 , and (3) ⁽²⁾ it follows

Theorem 6. The totality of all Riemannian spaces $\bar{V}_n$ admitting a nontrivial geodesic mapping onto V_n , distinct from an Einstein space, depends on $r \leq n(n+1)/2 - (n-2)$ essential parameters.

3. Finally, the displacement vector ξ^i of a nontrivial infinitesimal geodesic transformation of V_n satisfies the equations ⁽³⁾

$$h_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{kj} + \psi_j g_{ki},$$

where $h_{ij} = \xi_{i,j} + \xi_{j,i}$, $\psi_i \neq 0$ is a gradient vector. Therefore the tensor $\hat{g}_{ij} = h_{ij} - 2\psi g_{ij}$ satisfies the equations A_1 , and V_n admits a nontrivial geodesic mapping. On this basis, Theorems 5 and 6 give an estimate of the number of essential parameters on which the displacement vector of a general nontrivial infinitesimal geodesic transformation of V_n depends. Namely, in the first case $\tilde{r} \leq n^2 + 4$, and in the second $\tilde{r} \leq n^2 + 2$ (cf. ⁽⁴⁾).

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Note: Figure translations are in progress. See original paper for figures.

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