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Abstract

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MATHEMATICS

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THE STRUCTURE OF FUNDAMENTAL MATRICES OF R -SYSTEMS WITH ALMOST PERIODIC COEFFICIENTS

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For systems with periodic coefficients, the structure of fundamental matrices is described by Floquet's theorem (see, for example, ⁽¹⁾, p. 90, Theorem 5.1). A generalization of Floquet's theorem to systems with quasiperiodic coefficients was made by A. E. Gelman (see ^(7,8)) (the case $n = 2$) and by L. Ya. Adrianova (see ⁽²⁾) (the case of arbitrary n). Systems with almost periodic coefficients under certain special assumptions were studied by Lillo (see ⁽⁹⁾) and by B. F. Bylov (see ⁽³⁻⁶⁾).

In the present work a theorem is proved which generalizes Floquet's theorem to R -systems with almost periodic coefficients (uniformly almost periodic, see ⁽¹⁰⁾). For the proof, the method of ⁽¹³⁾ is used; moreover, the first part of the present work is devoted to the development of this method, and concerns arbitrary R -systems. The above-mentioned results and methods of reasoning of B. F. Bylov are also used.

Definition of an R -system. 1. Let the function $q(t)$ satisfy the Lipschitz condition. For each $\varepsilon > 0$ construct inductively T_i ($i = 1, 2, \dots$) as follows: $T_0 = 1$, $T_i = \sup$ of the lengths of intervals not containing intervals $[\tau_1, \tau_2]$ such that $\tau_2 - \tau_1 \geq T_{i-1}$ and

$$\frac{q(\tau_2) - q(\tau_1)}{\tau_2 - \tau_1} \leq \lim_{t \rightarrow \tau} \frac{q(t) - q(\tau)}{t - \tau} - \varepsilon.$$

We shall say that: a) $q(t) \in R_1$, if either some $T_i = +\infty$, or

$$\sum_{i=1}^{\infty} \frac{T_{i-1}}{T_i} = +\infty \quad (\varepsilon > 0 \text{ arbitrary});$$

b) $q(t) \in \widetilde{R}$, if $q(t) \in R_1$, $-q(t) \in R_1$.

2. $A(t) \in R$, if the following holds. Let $U^{(k)}(t)$ ($k = 1, 2, \dots$) be arbitrary Perron transformations which reduce the system $\dot{x} = A(t)x$ to triangular form $\dot{u} = B^{(k)}(t)u$, and let $t_k \rightarrow +\infty$ be an arbitrary sequence such that there exists, uniformly on intervals, the limit

$$q_i(t) = \lim_{k \rightarrow \infty} \int_{t_k}^{t_k+t} b_{ii}^{(k)}(\tau) d\tau \quad (b_{ii}^{(k)}(t) \text{ is the } i\text{-th diagonal element of } B^{(k)}(t)).$$

Then $q_i(t) \in \widetilde{R}$.

Lemma 1. *Let there be given a sequence of functions $q_k(t) \equiv q(t) \in \widetilde{R}$ ($t \geq t_0$) such that*

$$|q_k(t) - q_k(\tau)| \leq a(t - \tau) \quad (k = 1, 2, \dots; t \geq \tau \geq t_0).$$

Let

$$\lambda = \lim_{k \rightarrow \infty} \frac{q_k(t_k) - q_k(\tau_k)}{t_k - \tau_k}$$

for some sequence of intervals $[\tau_k, t_k]$, $t_k - \tau_k \xrightarrow[k \rightarrow \infty]{} +\infty$.

Then for every $\varepsilon > 0$ there exist a sequence $\theta_i \geq t_0$ and a sequence of indices k_i ($i = 1, 2, \dots$) such that

$$r_i(t) = q_{k_i}(\theta_i + t) - q_{k_i}(\theta_i) \longrightarrow q(t) \quad i \rightarrow \infty$$

uniformly on each interval for $t \geq 0$, and for the function $q(t)$ one has

$$q(t) - q(\tau) \geq (\lambda - \varepsilon)(t - \tau) - d_\varepsilon$$

for any $t \geq \tau \geq 0$ and some $d_\varepsilon \geq 0$ (d_ε is a function of ε).

We give the main idea of the proof. Suppose this is not so. Then for some $\varepsilon_0 > 0$ and some k one can find an interval L of the line on which the function $q_k(t)$ has mean increment $> \lambda - \varepsilon_0/2$; but on this interval there will be nonintersecting intervals whose union has relative (on L) measure close to 1, and on each of the intervals $q_k(t)$ has mean increment $< \lambda - \varepsilon_0$. This leads to a contradiction, which proves the lemma.

With the aid of Lemma 1 the following is proved.

Lemma 2. Let the vector-functions

$$q_k(t) = \{q_k^{(1)}(t), \dots, q_k^{(n)}(t)\} \quad (k = 1, 2, \dots)$$

be such that

$$\|q_k(t) - q_k(\tau)\| \leq a(t - \tau) \quad (k = 1, 2, \dots; t \geq \tau \geq t_0).$$

Let $\tau_k \geq t_0$ ($k = 1, 2, \dots$) and

$$q(t) = \{q^{(1)}(t), \dots, q^{(n)}(t)\} = \lim_{k \rightarrow \infty} [q_k(\tau_k + t) - q_k(\tau_k)]$$

(the limit is uniform on each interval for $t \geq 0$). Suppose that for any τ_k , $q^{(i)}(t) \in \bar{R}$.

Denote

$$\lambda = \overline{\lim}_{t \rightarrow +\infty} \frac{q^{(1)}(t) - q^{(1)}(\tau)}{t - \tau}.$$

Then for every $\eta > 0$ there exist numbers $\lambda_1, \dots, \lambda_n$ ($|\lambda_1 - \lambda| < \eta$), numbers $\theta_j \geq t_0$, and indices k_j ($j = 1, 2, \dots$) such that there exists

$$r(t) = \lim_{j \rightarrow \infty} [q_{k_j}(\theta_j + t) - q_{k_j}(\theta_j)]$$

(the limit is uniform on each interval for $t \geq 0$), and for every $i = 1, 2, \dots, n$

$$(\lambda_i - \varepsilon)(t - \tau) - d_\varepsilon \leq r^{(i)}(t) - r^{(i)}(\tau) \leq (\lambda_i + \varepsilon)(t - \tau) + d_\varepsilon$$

for every $\varepsilon > 0$, some $d_\varepsilon \geq 0$ (d_ε is a function of ε), and all $t \geq \tau \geq 0$.

We consider the systems

$$\dot{x} = A(t)x \text{ in } E^n; \quad (\|A(t)\| \leq a; t \geq t_0); \quad A(t) \in R; \quad (\text{I})$$

$$\dot{y} = A(t)y + \varphi(y, t); \quad \|\varphi(y, t)\| \leq g(t)\|y\| \quad (\text{II})$$

($A(t)$ and $\varphi(y, t)$ are continuous in t and in y).

Definition 1. The **maximal exponent** of the vector-function $x(t)$ will be called

$$\bar{\lambda} = \overline{\lim}_{t \rightarrow +\infty} \frac{1}{t - \tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|}.$$

The **minimal exponent** of $x(t)$ is defined as

$$\underline{\lambda} = \lim_{t \rightarrow +\infty} \frac{1}{t - \tau} \ln \frac{\|x(t)\|}{\|x(\tau)\|}.$$

Definition 2. A number λ will be called a **rough exponent** of the system (I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that from

$$g(t) = g_1(t) + g_2(t); \quad g_1(t) < \delta; \quad \int_{\tau}^{+\infty} g_2(\tau) d\tau < \infty$$

it follows that the system (II) has a generalized (i.e., ordinary or a shift of an ordinary or a limiting—see ⁽¹¹⁾) solution whose characteristic exponent $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$. The set of rough exponents of the system (I) will be called the **rough real spectrum** Λ_s of the system (I). (It is convenient to replace by this definition the definition of the corresponding concept given in ⁽¹³⁾.)

Theorem 1. For every generalized solution $x(t)$ of the system (I),

$$\bar{\lambda} \in \Lambda_s \quad \text{and} \quad \underline{\lambda} \in \Lambda_s \quad (A(t) \in R).$$

The proof is obtained by combining two results: Theorem 2 of note ⁽¹³⁾ and Lemma 2 of the present note.

We now pass to the case where $A(t)$ is an almost periodic matrix for $-\infty < t < +\infty$.

Theorem 2. Let $A(t)$ be an almost periodic matrix ($A(t) \in R$). Then there exists a Perron transformation $x = U(t)u$ reducing the system (I) to triangular form

$$\dot{u} = \begin{pmatrix} b_{11}(t), \dots, b_{1n}(t) & & \\ & 0 & \ddots \\ & & \ddots & \vdots \\ & & & b_{nn}(t) \end{pmatrix} u, \quad (1)$$

where the diagonal coefficients $b_{ii}(t)$ are “integrally close” to certain constants λ_i :

$$(\lambda_i - \varepsilon)(t - \tau) - d_\varepsilon \leq \int_{\tau}^t b_{ii}(\xi) d\xi \leq (\lambda_i + \varepsilon)(t - \tau) + d_\varepsilon, \quad (2)$$

$i = 1, 2, \dots, n$ (for every $\varepsilon > 0$, some $d_\varepsilon \geq 0$ (d_ε is a function of ε), and all $t \geq \tau$).

We outline the proof. Reduce the system (I) by a Perron transformation $U_1(t)$ to triangular form. It can be shown that the uniform continuity of $A(t)$ on

the line implies that $U_1(t)$ is uniformly continuous on the line. Using this and Lemma 2, we obtain that there exists a limiting system

$$\dot{x} = A^*(t)x, \quad \text{where} \quad A^*(t) = \lim_{t_k \rightarrow +\infty} A(t_k + t),$$

which is reduced by the Perron transformation $\widetilde{U}_1(t)$ to the form (1)–(2), with $\widetilde{U}_1(t)$ uniformly continuous on the line. Making use of the fact that

$$A(t) = \lim_{k \rightarrow +\infty} A^*(\theta_k + t),$$

we obtain that the system (I) itself is reduced by a Perron transformation to (1)–(2).

Theorem 3. *Let the system (I) have n distinct characteristic exponents $\lambda_1, \lambda_2, \dots, \lambda_n$ ($A(t) \in R$ is an almost periodic matrix).*

Then every fundamental matrix $X(t)$ of system (1) has the form

$$X(t) = S(t) \exp \left[\text{diag} \left\{ \int_0^t p_1(\tau) d\tau, \dots, \int_0^t p_n(\tau) d\tau \right\} \right],$$

where $S(t)$ is a Lyapunov almost-periodic matrix; the function $p_i(t)$ ($i = 1, 2, \dots, n$) is almost periodic and has mean value equal to λ_i .

Proof. By means of Theorem 2 and the methods of the papers ^(12,13), we obtain that the hypotheses of B. F. Bylov's theorem are satisfied (see ⁽³⁾, Theorem 2). With the aid of this theorem we obtain our assertion.

Generalizing the arguments of B. F. Bylov (see ⁽³⁾), by means of Theorem 2 we obtain, in the general case (i.e., when system (1) need not have n distinct characteristic exponents), the following theorem (generalizing Floquet's theorem).

Theorem 4. Let $A(t) \in R$ be an almost-periodic matrix, and let $x_1(t), \dots, x_n(t)$ be a normal system of solutions of system (1), the characteristic exponent of the solution $x_i(t)$ being equal to λ_i .

Then there exists a Lyapunov almost-periodic transformation $x = S(t)u$ reducing system (1) to the block-diagonal form

$$\dot{u} = \begin{pmatrix} B^{(1)}(t) & 0 \\ 0 & B^{(m)}(t) \end{pmatrix} u,$$

where each system

$$\dot{v}_k = B^{(k)}(t)v_k$$

is such that the logarithmic derivative of the norm of each of its solutions is “integrally close” to one of the constants λ_i (the same one for the given k).

Remark 1. From the theorems proved it follows that, for an R -system with almost-periodic coefficients, $\Lambda_s = \Lambda^0$, where Λ^0 is the set of characteristic exponents of the ordinary solutions of the system.

Remark 2. The theorems proved give a method for computing the characteristic exponents of R -systems with almost-periodic coefficients, completely analogous to that given by Floquet’s theorem for systems with periodic coefficients.

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