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Abstract

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MATHEMATICS

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ON RINGS OF OPERATORS IN A SPACE WITH INDEFINITE METRIC

(Presented by Academician L. S. Pontryagin on 25 I 1966)

A well-known theorem of L. S. Pontryagin asserts that a self-adjoint operator in a space of type Π_n ($0 < n < \infty$) has an n -dimensional nonnegative invariant subspace ⁽¹⁾. This result was extended by M. A. Naimark to a family of mutually commuting normal operators ⁽²⁾. In the present article a generalization of these results is obtained for symmetric representations in a space of type Π_n of rings with involution having a complete system of representations whose dimensions do not exceed the number $m < \infty$. Another (very simple) proof of L. S. Pontryagin's theorem is also obtained.

Theorem 1. Let T_x be a symmetric representation in a space H of type Π_n of a ring R with involution, having a complete system of representations whose dimensions do not exceed $m < \infty$. Then in H there exists a subspace H_1 , invariant with respect to all T_x ($x \in R$), such that $\dim H_1 \leq mn$, and the subspace $H_2 = \{x : (x, y) = 0 \text{ for all } y \in H_1\}$ is nonnegative.

Theorem 2. An irreducible ring of operators in a space H of type Π_n , closed in the strong topology and containing, together with each operator, also its adjoint (in the indefinite metric), coincides with the ring of all bounded operators in H .

We pass to the proof.

Lemma 1*. Let C be a family of bounded operators in a space H of type Π_n , and suppose that from $A \in C$ it follows that $A^* \in C$, and that C has no null invariant subspaces in H . Then

$$H = H^0 \oplus H^1 \oplus \dots \oplus H^m, \quad (1)$$

where H^0 is positive, H^i ($1 \leq i \leq m$) is either a space of type Π_{k_i} or a k_i -dimensional negative subspace, with $\sum k_i = n$; moreover $AH_j \subseteq H_j$ ($A \in C$, $0 \leq j \leq m$), and the restriction of C to H^i ($1 \leq i \leq m$) is irreducible. The summand H^0 in (1) may be absent.

Proof. Note that, under the conditions of the lemma, the family C cannot have degenerate invariant subspaces. (Indeed, if H_0 is a degenerate invariant

subspace, then $H_{0,0} = \{x : x \in H_0, (x, y) = 0 \text{ for all } y \in H_0\}$ will be an invariant null subspace.)

Suppose that there exist positive invariant subspaces in H . The set L of such subspaces is partially ordered by inclusion. If $\{H_\gamma\}$ is a chain in L , then the subspace $\mathcal{H} = \bigcup H_\alpha$ is invariant with respect to C and nonnegative and, by the preceding remark, nondegenerate; hence $\mathcal{H} \in L$, so that the chain $\{H_\gamma\}$ has an upper bound $\mathcal{H}$. By Zorn's lemma, there exists in H a maximal positive invariant subspace H^0 .

* This lemma was obtained by the author in the article ⁽⁶⁾; at the same time an analogous assertion was obtained by M. A. Naimark.

In the subspace $H_1 = H \oplus H^0$ (invariant with respect to C) there are neither positive nor degenerate subspaces. Hence it follows that C in H_1 is either irreducible, or there is a decomposition $H_1 = H_{1,1} \oplus H_{1,2}$, where $H_{1,i}$ ($i = 1, 2$) is invariant and is either a space of type Π_{s_i} or an s_i -dimensional negative subspace, with $0 < s_i < n$, $s_1 + s_2 = n$. If the restriction of C to $H_{1,i}$ ($i = 1, 2$) is reducible, then analogous reasoning leads to a decomposition $H_{1,i} = H_{1,i,1} \oplus H_{1,i,2}$.

Continuing this reasoning, we arrive (in no more than n steps) at the required decomposition.

Lemma 2. *A self-adjoint operator A in a space H of type Π_n ($0 < n < \infty$) has a nontrivial invariant subspace.*

Proof. If (x, y) is the indefinite scalar product in H , then

$$(x, y) = [(E - 2P)x, y],$$

where $[x, y]$ is the usual scalar product in H , and P is a projection operator, with $\dim PH = n$.

If $A^* = A$ in the Π_n -metric and A^+ is the operator adjoint to A with respect to the scalar product $[x, y]$, then, obviously,

$$A^+ - A = 2A^*P - 2PA,$$

whence it follows that the operator $\text{Im } A = i/2(A^+ - A)$ is finite-dimensional; it is known ⁽³⁾ that such an operator has a nontrivial invariant subspace.

The following lemma is equivalent to L. S. Pontryagin's theorem.

Lemma 3. *A self-adjoint operator A in a space H of type Π_n has a nonpositive eigenvector.*

Proof. If A has a zero invariant subspace $\mathcal{H}_0$, then $\dim \mathcal{H}_0 \leq n$, and the assertion of the lemma is obvious. If there are no such subspaces in H , then H admits the decomposition indicated in Lemma 1, and the restriction of A to

H_i ($1 \leq i \leq m$) has no nontrivial invariant subspaces. But then, by Lemma 2, $\dim H_i = 1$, as was required.

Lemma 4. *If C is an irreducible set of operators in a space H of type Π_n , and from $A \in C$ it follows that $A^* \in C$, and the operator B commutes with all $A \in C$, then $B = \lambda E$.*

Proof. We may assume that $B = B^*$ (otherwise we consider the operators $B + B^*$, $i(B - B^*)$). If λ is an eigenvalue of B (see Lemma 3) and $H_\lambda = \{x : Bx = \lambda x\}$, then, by the conditions of the lemma, $AH_\lambda \subseteq H_\lambda$ ($A \in C$). By the irreducibility of C , $H_\lambda = H$, i.e. $B = \lambda E$, as was required.

Proof of Theorem 2. Let C be an irreducible ring of operators in H , and suppose that from $A \in C$ it follows that $A^* \in C$, B is a bounded operator in H , $\{f_j^0, 1 \leq j \leq N\}$ are elements of H , and $\varepsilon > 0$. We shall show that there exists an $A \in C$ such that

$$\|Af_i^0 - Bf_i^0\| < \varepsilon \quad (1 \leq i \leq N).$$

Obviously, we may assume that the vectors f_j^0 are linearly independent.

Let $\mathcal{H} = H \oplus \dots \oplus H$ (N summands). In a natural way $\mathcal{H}$ becomes a Π_{nN} -space. Further, let P_i be the operator acting by the formula

$$P_i(f_1 \dots f_N) = f_i.$$

The operators

$$B'(f_1 \dots f_N) = (Bf_1, \dots, Bf_N)$$

form in $\mathcal{H}$ a ring C' with a natural involution.

Let $\mathcal{H}_0$ be the closure in $\mathcal{H}$ of the set of vectors of the form

$$(Bf_1^0, \dots, Bf_N^0) \quad (B \in C). \quad (2)$$

Obviously, $B'\mathcal{H}_0 \subseteq \mathcal{H}_0$ ($B' \in C'$), and $\mathcal{H}_0$ is a nondegenerate subspace (otherwise the subspace $\mathcal{H}_0^0 = \{x : x \in \mathcal{H}_0, (x, y) = 0 \text{ for all } y \in \mathcal{H}_0\}$ would be invariant with respect to $B' \in C'$ and $0 < \dim \mathcal{H}_0^0 \leq nN$; but then the subspace $P_i\mathcal{H}_0^0 \subset H$ would be invariant with respect to C and finite-dimensional for all $1 \leq i \leq N$, whence $P_i\mathcal{H}_0^0 = 0$, i.e. $\mathcal{H}_0^0 = 0$).

Define in $\mathcal{H}$ the operator Q as follows:

$$Qx = \begin{cases} x, & x \in \mathcal{H}_0, \\ 0, & x \in \mathcal{H} \ominus \mathcal{H}_0. \end{cases}$$

Then $QA = AQ$ ($A \in C$). Let $\{Q_{ij}, 1 \leq i, j \leq N\}$ be the matrix corresponding to Q . Clearly, $Q_{ij}A = AQ_{ij}$ ($A \in C$), whence, by Lemma 4, $Q_{ij} = \lambda_{ij}E$. At the same time

$$Q(f_1^0 \dots f_N^0) = (f_1^0 \dots f_N^0),$$

i.e. $\sum \lambda_{ij} f_j = f_i$ ($1 \leq i \leq N$), whence, by the linear independence of $\{f_j\}$, $\lambda_{ij} = 0$ for $i \neq j$, $\lambda_{ii} = 1$, i.e. $Q = E$. This means that the system of vectors of the form (2) is dense in $\mathcal{H}$, which proves the theorem.

Proof of Theorem 1. We shall use the following result of Godement (4): if a ring R has a complete system of representations of dimension $\leq m$, then every completely irreducible representation of R has dimension $\leq m$. Complete irreducibility means that every bounded operator in the representation space is a strong limit of operators T_x ($x \in R$). However, Godement's proof remains valid if the condition of complete irreducibility is replaced a priori by the weaker one: the strong closure of the ring $\{T_x\}$ coincides with the ring of all operators in the representation space.

Returning to Theorem 1, suppose first that there are no null invariant subspaces in H . Then the decomposition indicated in Lemma 1 holds, and, by Godement's theorem and Theorem 2, $\dim H_i \leq m$, which proves the theorem.

If there are null invariant subspaces in H , and H_0 is some maximal one among them, then $\dim H_0 = k \leq N$ and the subspace

$$H_0^\perp \{x \in H, x \perp H_0\}$$

is invariant with respect to T_x ($x \in R$), with $H_0 \subset H_0^\perp$. The quotient space

$$\widehat{H}_0 = H_0^\perp / H_0$$

is naturally transformed into a space of type Π_{n-k} (5), and in $\widehat{H}_0$ there arises a representation $\widehat{T}_x$ that no longer contains null invariant subspaces. But then in $\widehat{H}_0$ there exists an invariant subspace of dimension $\leq m(n-k)$. Thus in $H_0^\perp$ there exists an invariant subspace of dimension

$$k + (n-k)m \leq nm,$$

satisfying the requirements of Theorem 1.

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