

ON THE APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY ALGEBRAIC POLYNOMIALS ON AN INTERVAL

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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF CONTINUOUS AND DIFFERENTIABLE FUNCTIONS BY ALGEBRAIC POLYNOMIALS ON AN INTERVAL

(Presented by Academician A. N. Kolmogorov, May 15, 1965)

1. Let $E_n(f)$ be the best uniform approximation of a function $f(x)$, continuous on the interval $[-1, 1]$, by algebraic polynomials of degree not exceeding n . If $\mathfrak{M}$ is some class of functions, then we set

$$E_n[\mathfrak{M}] = \sup_{f \in \mathfrak{M}} E_n(f) \quad (n = 0, 1, 2, \dots).$$

In the case of approximation of 2π -periodic functions by trigonometric polynomials of order $\leq n$ on the whole axis, we shall write correspondingly $E_n^*(f)$, $E_n^*[\mathfrak{M}]$.

We denote by $W^{(r)}H_\omega$ ($r = 0, 1, 2, \dots$, $W^{(0)}H_\omega = H_\omega$) the class of functions $f(x)$, defined on the interval $[-1, 1]$, for which the modulus of continuity $\omega(f^{(r)}; t)$ of the r -th derivative does not exceed a prescribed modulus of continuity $\omega(t)$. If $\omega(t) = Kt^\alpha$ ($0 < \alpha \leq 1$), we shall write $W^{(r)}KH^{(\alpha)}$. The corresponding classes of 2π -periodic functions will be denoted by $W^{(r)}H_\omega^*$, $W^{(r)}KH^{(\alpha)*}$.

Exact estimates of the quantities $E_n^*[W^{(r)}H_\omega^*]$, obtained in papers ^(1,2) for the case $\omega(t) = Kt$, $r = 0, 1, 2, \dots$, and in papers ⁽³⁻⁶⁾ for an arbitrary concave upward modulus of continuity $\omega(t)$ and $r = 0, 1, 2, 3$, can be written in the form

$$E_{n-1}^*[W^{(r)}H_\omega^*] = E_{n-1}^*(f_{nr}) = \max_x |f_{nr}(x)| \quad (n = 1, 2, \dots), \quad (1)$$

where $f_{nr}(x) = f_{nr}(\omega; x)$ is a function of period $2\pi/n$ with mean value over a period equal to zero, whose derivative of order r ($r = 0, 1, 2, \dots$) is defined by the equalities

$$f_{nr}^{(r)}(x) = \begin{cases} \frac{1}{2} \omega(2x), & (0 \leq x \leq \pi/2n), \\ -\frac{1}{2} \omega(-2x), & (-\pi/2n \leq x \leq 0), \end{cases}$$

$$f_{nr}^{(r)}(x + \pi/n) = -f_{nr}^{(r)}(x).$$

Theorem 1. *For any concave upward modulus of continuity $\omega(t)$ and for each $r = 0, 1, 2, \dots$, there exists a sequence of functions $g_n(x) \in W^{(r)}H_\omega$ such that*

$$E_{n-1}(g_n) \geq (1 - \varepsilon_n) \max |f_{nr}(x)|,$$

where $\varepsilon_n \geq 0$, $\varepsilon_n = O(1/\ln n)$.

For the class $KH^{(1)}$ this assertion was proved earlier by S. M. Nikol'skii⁽⁷⁾. By means of Theorem 1 one can show that, for arbitrary concave $\omega(t)$ and $r = 0, 1, 2, 3$, the upper bound $E_{n-1}[W^{(r)}H_\omega]$ is asymptotically equal to the right-hand side of equality (1). The fact of asymptotic coincidence of the quantities $E_n[W^{(r)}KH^{(\alpha)}]$ and $E_n^*[W^{(r)}KH_*^{(\alpha)}]$ ($r = 0, 1, 2, \dots$) was established in papers⁽⁷⁻⁹⁾.

2. S. M. Nikol'skii, in 1946, was the first to discover one important feature of approximation of functions on an interval. He showed⁽⁷⁾ that for any function $f \in KH^{(1)}$ one can construct a sequence of algebraic polynomials $P_n(f; x)$ of degree $\leq n$ (depending linearly on f) such that, for all $x \in [-1, 1]$,

$$\begin{aligned} |f(x) - P_{n-1}(f; x)| &\leq \frac{K\pi}{2n} \sqrt{1-x^2} + |x| O(\ln n/n^2) \\ &= E_{n-1}[KH^{(1)}] \{\sqrt{1-x^2} + o(1)\}, \end{aligned}$$

and this relation holds uniformly with respect to x . Thus the polynomials $P_n(f; x)$, while giving asymptotically best approximation on the class $KH^{(1)}$, give a significantly smaller deviation at the ends of the interval $[-1, 1]$. Later this result was generalized⁽¹⁰⁾ to the classes $W^{(r)}KH^{(1)}$ ($r = 1, 2, \dots$).

It turns out that an analogous fact also holds for other, broader, classes of functions.

Theorem 2. *For any function $f \in KH^{(\alpha)}$ ($0 < \alpha < 1$) there exists a sequence of algebraic polynomials $P_n(f; x)$ of degree $\leq n$ such that, uniformly in $x \in [-1, 1]$, the inequality*

$$\begin{aligned} |f(x) - P_{n-1}(f; x)| &\leq \frac{K}{2} \left(\frac{\pi}{n} \sqrt{1-x^2} \right)^\alpha + O(n^{-3/2\alpha}) \\ &= E_{n-1}[KH^{(\alpha)}] \{(1-x^2)^{\alpha/2} + o(1)\}. \end{aligned} \quad (2)$$

Theorem 3. *Whatever convex modulus of continuity $\omega(t)$ may be, for any function $f \in H_\omega$ there exists a sequence of algebraic polynomials $P_n(f; x)$ of degree $\leq n$ such that, uniformly in $x \in [-1, 1]$,*

$$|f(x) - P_{n-1}(f; x)| \leq \frac{1}{2} \omega\left(\frac{\pi}{n} \sqrt{1-x^2}\right) + o\left(\omega\left(\frac{1}{n}\right)\right). \quad (3)$$

The constant 1/2 on the right-hand side cannot be decreased.

Theorem 4. *Whatever convex modulus of continuity $\omega(t)$ may be, for any function $f \in W^{(1)}H_\omega$ there exists a sequence of algebraic polynomials $P_n(f; x)$ of degree $\leq n$ such that, uniformly in $x \in [-1, 1]$,*

$$|f(x) - P_{n-1}(f; x)| \leq \frac{1}{4} \int_0^{\frac{\pi}{n} \sqrt{1-x^2}} \omega(t) dt + o\left(\frac{1}{n} \omega\left(\frac{1}{n}\right)\right). \quad (4)$$

The constant 1/4 on the right-hand side cannot be decreased.

Let us note that the polynomials $P_n(f; x)$ referred to in Theorems 2-4 depend on the function $f(x)$ nonlinearly.

The proof of Theorems 2-4 is based on the idea of intermediate approximation of functions of the classes H_ω (respectively $W^{(1)}H_\omega$) by continuous functions having almost everywhere a first (respectively, second) derivative bounded by a certain majorant depending on x . In this way asymptotically exact estimates are also obtained. For example, the following assertion holds.

Theorem 5. *If $\omega(t)$ is an arbitrary convex modulus of continuity, then for any function $f \in H_\omega$ one can specify a sequence of polygonal functions $\varphi_n(x)$ ($n = 1, 2, \dots$) such that:*

1) *for almost all $x \in [-1, 1]$,*

$$|\varphi'_n(x)| \leq M_n(x) = \frac{1}{2} \left\{ \omega'_+ \left(\frac{\pi}{n} \sqrt{1-x^2} \right) + \omega'_- \left(\frac{\pi}{n} \sqrt{1-x^2} \right) \right\} \quad (n = 1, 2, \dots);$$

* The right-hand sides of inequalities (3) and (4) asymptotically coincide respectively with $E_{n-1}[H_\omega]$ and $E_{n-1}[W^{(1)}H_\omega]$ for $x = 0$.

2) *at each point x of the interval $[-1, 1]$, uniformly in x ,*

$$|f(x) - \varphi_n(x)| \leq \frac{1}{2} \max_{0 \leq t \leq 2} [\omega(t) - tM_n(x)] + o\left(\omega\left(\frac{1}{n}\right)\right).$$

3. In paper (11) it is proved that, for any function $f(x)$ continuous on the interval $[-1, 1]$, the inequalities

$$E_{n-1}(f) \leq \omega(f; \pi/n) \quad (n = 1, 2, \dots). \quad (5)$$

hold. From analogous considerations and Theorem 3 it follows that there exists a sequence of polynomials $P_n(f; x)$ such that

$$|f(x) - P_{n-1}(f; x)| \leq \omega\left(f; \frac{\pi}{n} \sqrt{1-x^2}\right) + o\left(\omega\left(f; \frac{1}{n}\right)\right) \quad (-1 \leq x \leq 1). \quad (6)$$

Estimates (5) and (6) are, as the following theorem shows, in a certain sense unimprovable.

Theorem 6. Whatever the number $\delta > 0$, there exists a sequence $f_n(x)$ of functions continuous on $[-1, 1]$ for which

$$E_{n-1}(f_n) \geq (1 - \varepsilon_n) \omega\left(f_n; \frac{\pi - \delta}{n}\right),$$

where $\varepsilon_n \geq 0$, $\varepsilon_n = O(1/\ln n)$.

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