

# RESTRICTION OF A REPRESENTATION OF THE COMPLEMENTARY SERIES OF THE COMPLEX UNIMODULAR GROUP OF SECOND ORDER TO A REAL SUBGROUP

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**Abstract**

**Full Text**

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*MATHEMATICS*

**B. D. ROMM**

# RESTRICTION OF A REPRESENTATION OF THE COMPLEMENTARY SERIES OF THE COMPLEX UNIMODULAR GROUP OF SECOND ORDER TO A REAL SUBGROUP

*(Presented by Academician I. M. Vinogradov, 16 IX 1965)*

I. In paper <sup>(6)</sup>, representations of the principal series of the group  $G_0$  of complex matrices of second order with determinant equal to 1 are considered. If these representations are restricted to the real subgroup  $G$ , then reducible representations arise, which there decompose into irreducible ones. In the present article a similar problem is solved for the complementary series.

As is known <sup>(1)</sup>, a representation of the complementary series is determined by a parameter  $\sigma$ ,  $0 < \sigma < 2$ , and is realized in a Hilbert space  $\mathfrak{H}_\sigma$ , which is the closure of the set of functions  $f(z)$  on the complex plane satisfying the condition

$$(f, f) = \int |z_1 - z_2|^{-2+\sigma} f(z_1) \overline{f(z_2)} dz_1 dz_2 < \infty, \quad (1)$$

where  $dz = dx dy$ ,  $x = \operatorname{Re} z$ ,  $y = \operatorname{Im} z$ . The operators  $V_a$  of the representation have the form

$$V_a f(z) = |\beta z + \delta|^{-2-\sigma} f(z\tilde{a}), \quad (2)$$

where

$$a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad z\tilde{a} = \frac{\alpha z + \gamma}{\beta z + \delta}.$$

For the group  $G$  the operators  $V_a$  define a certain reducible unitary representation  $d$ . The problem of the present article is to decompose the representation  $d$  into irreducible representations.

II. Consider the subrepresentation  $d^+$  of the representation  $d$  in the subspace  $\mathfrak{H}_\sigma^+$  of elements  $f \in \mathfrak{H}_\sigma$  with support in the upper half-plane  $Z^+$ . The decomposition of  $d$  into irreducible representations reduces to the decomposition of the representations of the group  $G$  analogous to  $d^+$ . Therefore we first decompose the representation  $d^+$ .

Let  $\mathfrak{H} = \mathfrak{H}_\sigma^+ \cap L_2^\mu(Z^+) \cap L_1^{\mu'}(Z^+)$ , where the measures  $\mu(z)$  and  $\mu'(z)$  have the form  $d\mu(z) = (\text{Im } z)^\sigma dz$ ,  $d\mu'(z) = (\text{Im } z)^{\sigma/2-1} dz$ . Put, for  $z = i\tilde{a}$ ,  $a \in G$ ,

$$\psi(a) = (\text{Im } z)^{\sigma/2+1} f(z). \quad (3)$$

It is easy to see that for any  $a_0 \in G$

$$\psi(aa_0) = (\text{Im } z)^{\sigma/2+1} V_{a_0} f(z). \quad (4)$$

The totality of all matrices  $u_\varphi$

$$u_\varphi = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \quad (5)$$

forms a subgroup  $U$  in the group  $G$ . If the subgroup  $K \subset G$  consists of matrices

$$k = \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix}, \quad \lambda > 0,$$

then the unique decomposition holds:

$$a = uk, \quad (6)$$

where  $u \in U$ ,  $k \in K$ . For the Haar measure  $d\mu(a)$  on the group  $G$  we have

$$d\mu(a) = d\varphi dz / 2\pi(\text{Im } z)^2, \quad (7)$$

where  $z = i\tilde{a} = i\tilde{k} = i\lambda^2 + \lambda\mu$ .

If  $u_1 \in U$ , then  $\tilde{u}u_1 = i$ ; therefore from the decomposition (6) it follows that

$$\psi(ua) = \psi(a) = \psi(k).$$

From formulas (3) and (7) it follows that

$$\int |\psi(a)|^2 d\mu(a) = \int_{Z^+} |f(z)|^2 (\text{Im } z)^\sigma dz,$$

$$\int |\psi(a)| d\mu(a) = \int_{Z^+} |f(z)| (\operatorname{Im} z)^{\sigma/2-1} dz. \quad (9)$$

Thus,  $\psi(a) \in L = L_2(G) \cap L_1(G)$ , where the spaces  $L_p(G)$  are defined by the measure  $d\mu(a)$ , if in (3)  $f(z) \in \mathfrak{H}$ .

Let  $z = i\tilde{a}$ ,  $z_1 = i\tilde{a}_1$ , where  $a, a_1 \in G$ . Denote

$$\alpha_\sigma(a, a_1) = |z - z_1|^{-2+\sigma} / (\operatorname{Im} z \operatorname{Im} z_1)^{\sigma/2-1}. \quad (10)$$

According to (1), (3), and (7), for  $f(z) \in \mathfrak{H}$

$$(f, f) = \int \alpha_\sigma(a, a_1) \psi(a) \overline{\psi(a_1)} d\mu(a) d\mu(a_1), \quad (11)$$

$$\alpha_\sigma(aa_0, a_1a_0) = \alpha_\sigma(a, a_1), \quad a_0 \in G, \quad (12)$$

$$(f, f) = \int \alpha_\sigma(e, a_1) \int \psi(a) \overline{\psi(a_1a)} d\mu(a) d\mu(a_1), \quad (13)$$

where

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that if a sequence of compact sets  $Q_i$ ,  $i = 1, 2, \dots$ , in  $G$  satisfies the conditions  $Q_1 \subset Q_2 \subset \dots$  and  $\bigcup_i Q_i = G$ , then

$$(f, f) = \lim_{n \rightarrow \infty} \int_{Q_n} \alpha_\sigma(e, a_1) \int \psi(a) \overline{\psi(a_1a)} d\mu(a) d\mu(a_1), \quad (13a)$$

because an analogous equality holds for any summable function  $\varphi(a)$  on  $G$ :

$$\int \varphi(a) d\mu(a) = \lim_{n \rightarrow \infty} \int_{Q_n} \varphi(a) d\mu(a). \quad (13b)$$

Conversely, if  $\varphi(a)$  is a measurable function and the limit in the right-hand side of (13b) exists, then  $\varphi(a)$  is summable and (13b) holds.

III. Let  $a \rightarrow T_a$  be some irreducible unitary representation of the group  $G$ . Put

$$T_\psi = \int \psi(a) T_a^{-1} d\mu(a), \quad (14)$$

$\psi(a) \in L$ , because  $f(z) \in \mathfrak{H}$  according to (9);  $\Phi(b) = \int \psi(a) \overline{\psi(ba)} d\mu(a)$ . According to (8),  $\Phi(bu_1) = \Phi(b)$ ; therefore  $T_\Phi = T_u T_\Phi$  for any  $u \in U$ . It is clear that the inequality  $T_\Phi \neq 0$  is possible only in the case of a representation of class 1.

Of the principal series described in the article <sup>(2)</sup>, only the series  $C_q^0$  belongs to this class. It can be realized in the space  $L_2(-\infty, \infty)$  by the formula <sup>(4)</sup>

$$T_a f(x) = |\beta x + \delta|^{i\rho-1} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right), \quad a = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \quad (15)$$

The parameter  $\rho \in (-\infty, \infty)$  determines the representation.

In  $L_2(-\infty, \infty)$  there exists, unique up to a numerical factor, a normalized vector  $f_0(x)$  satisfying the condition

$$T_u f_0 = f_0. \quad (16)$$

for all  $u \in U$  (see (2), §5). In view of the one-dimensionality of the operator  $T_\Phi$  in the case of the series  $C_q^0$ ,

$$\text{Sp } T_\Phi = (T_\Phi f_0, f_0)_1, \quad (17)$$

where  $(f, f_1)_1$  is the scalar product in  $L_2(-\infty, \infty)$ , and has the form:

$$(f, f_1)_1 = \int_{-\infty}^{\infty} f(x) \overline{f_1(x)} dx.$$

According to the results of the article <sup>(4)</sup>

$$\begin{aligned} \int \psi(a) \overline{\psi(a_1 a)} d\mu(a) &= \frac{1}{32\pi^2} \int_0^\infty \rho \operatorname{th} \frac{\pi\rho}{2} (T_\psi^* T_\psi f_0, T_{a_1} f_0)_1 d\rho = \\ &= \frac{1}{32\pi^2} \int_0^\infty \rho \operatorname{th} \frac{\pi\rho}{2} (T_\psi f_0, T_\psi f_0)_1 (f_0, T_{a_1} f_0)_1 d\rho, \end{aligned} \quad (18)$$

because the space of vectors  $f_0$  satisfying condition (16) is one-dimensional, so that from equalities (8) and (14) it follows that  $T_\psi^* f_0 = T_u T_\psi^* f_0 = \nu f_0$ , where  $\nu = \text{const}$ , and the asterisk denotes passage to the adjoint operator.

The integral with respect to  $\rho$  in (18) converges uniformly in  $a_1$ ; therefore from equality (13a) we obtain

$$(f, f) = \lim_{n \rightarrow \infty} \frac{1}{32\pi^2} \int_0^\infty \int_{Q_n} \alpha_\sigma(e, a) (f_0, T_a f_0)_1 d\mu(a) \rho \operatorname{th} \frac{\pi\rho}{2} (T_\psi f_0, T_\psi f_0)_1 d\rho. \quad (19)$$

IV. For almost all  $a \in G$  there is a decomposition

$$a = u_\varphi \varepsilon u_{\varphi_1}, \quad (20)$$

where  $u_\varphi$  is determined by formula (5), and

$$\varepsilon = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

with  $t > 0$  ((<sup>4</sup>), appendix).  $a$  uniquely determines  $t, \varphi$ , and  $\varphi_1$ , if  $a \neq e$ ,  $\varphi, \varphi_1 \in [0, \pi)$ . In view of (16),  $\nu(a) = \nu(\varepsilon)$ , where  $\nu(a) = (f_0, T_a f_0)$ .

It is easy to show (<sup>5</sup>) that

$$\nu(\varepsilon) = \frac{1}{2} [P_{(-i\rho-1)/2}(\operatorname{ch} 2t) + P_{(i\rho-1)/2}(\operatorname{ch} 2t)], \quad (21)$$

where  $P_\nu(z)$  is the spherical Legendre function of the 1st kind.

From equalities (12) and (20) it follows that  $\alpha_\sigma(e, a) = |2 \operatorname{sh} t|^{-2+\sigma}$ . If we put  $y = \operatorname{ch} 2t$ , then equality (19) may be written in the form

$$(f, f) = \lim_{n \rightarrow \infty} \frac{2^{\sigma/2-7}}{\pi} \int_0^\infty \int_{\widetilde{Q}_n} (y-1)^{-1+\sigma/2} [P_{(i\rho-1)/2}(y) + P_{(-i\rho-1)/2}(y)] dy \rho \times \\ \times \operatorname{th} \frac{\pi\rho}{2} \int h(x, \rho) \overline{h(x, \rho)} dx d\rho, \quad (22)$$

where, according to (8) and (15),  $h(x, \rho) = T_\varphi f_0$ ,  $\widetilde{Q}_n = \{y : y = \operatorname{ch} 2t, a \in Q_n\}$ .

If we put

$$K(z, x, \rho) = \frac{1}{\sqrt{\pi}} (\operatorname{Im} z)^{(\sigma-1-i\rho)/2} |i(x - \operatorname{Re} z) - \operatorname{Im} z|^{i\rho-1},$$

then, according to (3) ( $K$  is the integral operator with kernel  $K(z, x, \rho)$ ),

$$h(x, \rho) = \int_{Z^+} K(z, x, \rho) f(z) dz = Kf(z), \quad f(z) \in \mathfrak{H}. \quad (23)$$

From the decomposition (20) it follows that  $a^2 + \beta^2 + \gamma^2 + \delta^2 = 2 \operatorname{ch} 2t$ , so that

$$\tilde{Q}_n = \left\{ y : y = \frac{a^2 + \beta^2 + \gamma^2 + \delta^2}{2}, a \in Q_n \right\}. \quad (24)$$

The inversion formula has the form

$$f(z) = \frac{1}{64\pi^2} (\operatorname{Im} z)^{-\sigma} \int \int_{-\infty}^{\infty} \rho \operatorname{th} \frac{\pi \rho}{2} \overline{K(z, x, \rho)} h(x, \rho) dx d\rho. \quad (25)$$

In view of (4) and (14), from the relation  $h = T_\psi f_0$  it follows that  $h(x, \rho)$  passes into  $T_{a_0} h(x, \rho)$  when  $f(z)$  passes into  $V_{a_0} f(z)$ .

**Theorem 1.** Let  $\mathfrak{H}_+$  denote the Hilbert space obtained by completing, in the scalar product

$$\begin{aligned} [h, h_1]_+ = \lim_{n \rightarrow \infty} \frac{2^{\sigma/2-7}}{\pi} \int_0^\infty \int_{\tilde{Q}_n} (y-1)^{-1+\sigma/2} [P_{(i\rho-1)/2}(y) + P_{(-i\rho-1)/2}^*(y)] dy \rho \\ \times \operatorname{th} \frac{\pi \rho}{2} \int h(x, \rho) \overline{h_1(x, \rho)} dx d\rho \end{aligned}$$

the set of all measurable functions  $h(x, \rho)$  for which

$$(h, h) = \iint |h(x, \rho)|^2 \rho \operatorname{th} \frac{\pi \rho}{2} dx d\rho < \infty$$

and the limit on the right-hand side of equality (22) exists. Then  $\mathfrak{H}_+$  is isometric to  $\mathfrak{H}_\sigma^+$ , and the isometry formula (22) is the Plancherel formula for the representation  $d^+$ . (23) and (25) are mutually inverse formulas, which hold when  $f(z) \in \mathfrak{H}$ .

V. Let now  $f(z) \in \mathfrak{H}_\sigma$ . Put  $\varphi(z) = (f(z) + f(\bar{z}))/2$ ,  $\varphi_1(z) = (f(z) - f(\bar{z}))/2$ .

**Theorem 2.** Let  $\mathfrak{H}(\mathfrak{H}_1)$  be the Hilbert space obtained by completing, in the scalar product  $[h, h_1]$  ( $[h, h_1]_1$ ),

$$[h, h_1] = \lim_{n \rightarrow \infty} \frac{2^{\sigma/2-6}}{\pi} \int_0^\infty \int_{\tilde{Q}_n} [(y-1)^{-1+\sigma/2} + (y+1)^{-1+\sigma/2}] [P_{(i\rho-1)/2}(y) +$$

$$+ P_{(-i\rho-1)/2}(y)] dy \rho \operatorname{th} \frac{\pi\rho}{2} \int h(x, \rho) \overline{h_1(x, \rho)} dx d\rho,$$

$$[h, h_1]_1 = \lim_{n \rightarrow \infty} \frac{2^{\sigma/2-6}}{\pi} \int_0^\infty \int_{\tilde{Q}_n} [(y-1)^{-1+\sigma/2} - (y+1)^{-1+\sigma/2}] [P_{(i\rho-1)/2}(y) + \\ + P_{(-i\rho-1)/2}(y)] dy \rho \operatorname{th} \frac{\pi\rho}{2} \int h(x, \rho) \overline{h_1(x, \rho)} dx d\rho$$

the set  $M(M_1)$  of all measurable functions  $h(x, \rho)$  for which  $(h, h) < \infty$ ,  $[h, h] < \infty$ ,  $((h, h) < \infty, [h, h]_1 < \infty)$ . Then the Plancherel formula for the representation  $d$  holds:

$$(f, f) = [h, h] + [h_1, h_1]_1,$$

where  $h(x, \rho) = K\varphi(z)$ ,  $h_1(x, \rho) = K\varphi_1(z)$ , and  $\varphi(z)$  and  $\varphi_1(z)$  belong to certain dense sets, which are defined analogously to  $\mathfrak{H}$ . There is an inversion formula of the form (25).

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*Note: Figure translations are in progress. See original paper for figures.*

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