

ON THE SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY ALGEBRAIC POLYNOMIALS

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Abstract

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MATHEMATICS

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ON THE SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVES BY ALGEBRAIC POLYNOMIALS

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The following assertion generalizes A. F. Timan's theorem on the approximation of a function given on a finite interval by algebraic polynomials ⁽¹⁾, to the case of simultaneous approximation of a function and its derivatives. It also strengthens the corresponding results of G. Freud ⁽²⁾ and A. O. Gelfond ⁽³⁾.

Theorem. For every function $f(x)$ having a continuous r -th derivative on the finite interval $[-1, 1]$, and for every natural number $n \geq r$, one can indicate an algebraic polynomial $Q_n(x)$ of degree not exceeding n such that, for all $s = 0, 1, 2, \dots, r$; $-1 \leq x \leq 1$,

$$|f^{(s)}(x) - Q_n^{(s)}(x)| \leq C_r \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right]^{r-s} \omega \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right],$$

where $\omega(\delta) = \omega(f^{(r)}; \delta)$; $\delta \in [0, \infty)$ is the modulus of continuity of $f^{(r)}(x)$; C_r is a constant depending only on r .

The polynomial $Q_n(x)$ can be written explicitly. To this end, consider the kernels $U_{N,s}(t)$ ($s = 0, 1, 2, \dots, r$; $N = 1, 2, \dots$)

$$U_{N,s}(t) = \frac{1}{\psi_N^{(s)}} \left(\frac{\sin Nt/2}{N \sin t/2} \right)^{2s+4},$$

where

$$\psi_N^{(s)} = \int_{-\pi}^{\pi} \left(\frac{\sin Nt/2}{N \sin t/2} \right)^{2s+4} dt.$$

Next put

$$P_{N,s}(f; x) = \int_{-\pi}^{\pi} f(\cos t) U_{N,s}(t - \arccos x) dt.$$

Let E be the identity operator, $N = [(n-r)/(r+2)] + 1$. Then

$$Q_n(x) = f(x) - (E - P_{N,r}) \left(\int_0^x (E - P_{N,r-1}) \times \right. \\ \left. \times \left(\int_0^{x_1} \dots \left(\int_0^{x_{r-2}} (E - P_{N,1}) \left(\int_0^{x_{r-1}} (E - P_{N,0})(f^{(r)}; x_r) dx_r \right) dx_{r-1} \right) \dots \right) dx_1 \right).$$

The main role in the proof of the theorem is played by the following

Lemma. *Let the function $f(x)$ have a continuous first derivative on the interval $[-1, 1]$, and suppose that*

$$|f'(x)| \leq \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right]^{\nu-1} \omega \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right]$$

($\omega(\delta)$ is some modulus of continuity, ν is a natural number). Then for $-1 \leq x \leq 1$ the inequalities

$$|f(x) - P_{n,\nu}(f; x)| \leq M_\nu \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right]^\nu \omega \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right],$$

$$|P_{n,\nu}^{(s)}(f; x)| \leq K_\nu \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right]^{\nu-s} \omega \left[\frac{1}{n} \left(\sqrt{1-x^2} + \frac{1}{n} \right) \right] \\ (s = 1, 2, \dots, \nu).$$

Omitting the proof of this lemma because of its bulkiness, we pass to the proof of the theorem.

Since for all $n \geq r$

$$\frac{1}{N} = \frac{1}{n} \cdot \frac{n}{1 + [(n-r)/(r+2)]} \leq 2(r+1) \frac{1}{n},$$

it is enough to verify the relations

$$|f^{(s)}(x) - Q_n^{(s)}(x)| \leq C_r \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{r-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right] \\ (s = 0, 1, 2, \dots, r; \quad -1 \leq x \leq 1).$$

First note that, by virtue of the assertion proved in (1),

$$|(E - P_{N,0})(f^{(r)}; x)| \leq A\omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right],$$

where $\omega(\delta) = \omega(f^{(r)}; \delta)$ and A is an absolute constant. The function

$$\Phi_1(x) = \int_0^x (E - P_{N,0})(f^{(r)}; t) dt$$

has on the interval $[-1, 1]$ a continuous first derivative, and moreover

$$|\Phi_1'(x)| \leq A\omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right].$$

By the lemma,

$$|(E - P_{N,1})(\Phi_1; x)| \leq AM_1 \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right] \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right],$$

$$|P'_{N,1}(\Phi_1; x)| \leq AK_1 \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right].$$

Hence, for $s = 0, 1$,

$$\left| \frac{d^s}{dx^s} (E - P_{N,1})(\Phi_1; x) \right| \leq C_1 \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{1-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right].$$

Now put

$$\Phi_2(x) = \int_0^x (E - P_{N,1})(\Phi_1; t) dt.$$

Analogously to the preceding, referring to the lemma, we obtain

$$|(E - P_{N,2})(\Phi_2; x)| \leq C_1 M_2 \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^2 \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right],$$

and for $s = 1, 2$

$$|P_{N,2}^{(s)}(\Phi_2; x)| \leq C_1 K_2 \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{2-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right].$$

Hence, for $s = 0, 1, 2$,

$$\left| \frac{d^s}{dx^s} (E - P_{N,2})(\Phi_2; x) \right| \leq C_2 \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{2-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right].$$

Continuing this process, each time setting

$$\Phi_\nu(x) = \int_0^x (E - P_{N,\nu-1})(\Phi_{\nu-1}; t) dt,$$

we finally obtain

$$\left| \frac{d^s}{dx^s} (E - P_{N,r})(\Phi_r; x) \right| \leq C_r \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]^{r-s} \omega \left[\frac{1}{N} \left(\sqrt{1-x^2} + \frac{1}{N} \right) \right]$$

$$(s = 0, 1, 2, \dots, r).$$

It remains to note that

$$(E - P_{N,r})(\Phi_r; x) = f(x) - Q_n(x).$$

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References Cited

¹ A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960, pp. 276-280. ² G. Szegő, *Orthogonal Polynomials*, Moscow, 1962, pp. 20-22. ³ A. O. Gelfond, UMN, 10, no. 1 (1955).

Note: Figure translations are in progress. See original paper for figures.

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