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Abstract

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MATHEMATICS

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THE MILNE PROBLEM WITH AN ARBITRARY PHASE FUNCTION AND TAKING ACCOUNT OF THE AZIMUTHAL INHOMOGENEITY OF THE SOLUTION

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The problem of the passage of radiation through a semi-infinite plane layer of matter (with inward normal to the boundary $\mathbf{n}$) leads to the equation

$$x(\tau, \vec{\omega}) = \hat{A}x(\tau, \vec{\omega}) + \mathcal{F}(\tau, \vec{\omega}), \quad 0 < \tau < \infty, \quad \vec{\omega} \in \Omega. \quad (1)$$

Here $x(\tau, \vec{\omega})$ is the phase density of the radiation at optical depth $\tau \in (0, \infty)$, propagating in the direction $\vec{\omega} \in \Omega$ (Ω is the unit sphere of three-dimensional space). The distribution of radiation sources and the boundary condition at $\tau = 0$ determine the form of $\mathcal{F}(\tau, \vec{\omega})$. The definition of the operator $\hat{A}$ will be given below.

We shall assume that the scattering phase function $g(\mu)$ satisfies conditions 1) of ⁽¹⁾. In addition, we adopt the notation of the paper ⁽¹⁾.

Let ε be a real number. Denote by B_ε the class of functions $x(\tau)$, defined on $(0, \infty)$ with values in $L_2(\Omega)$, such that: a) for $0 < \tau_1 < \infty$

$$\sup_{\tau \in (0, \tau_1)} \|x(\tau)\| < \infty;$$

- b) $e^{-\varepsilon\tau}x(\tau)$ is summable on $(0, \infty)$ in the Bochner sense ⁽²⁾. To distinguish a function from B_ε from its value $x(\tau) \in L_2(\Omega)$ at the point τ , we shall write $x(\cdot)$; $x(\tau, \vec{\omega})$ is the value assumed by the function $x(\tau) \in L_2(\Omega)$ at the point $\vec{\omega} \in \Omega$. If $x(\cdot) \in B_\varepsilon$, then $y(\tau) = \hat{g}x(\tau) \in C(\Omega)$ ⁽¹⁾, $\tau \in (0, \infty)$, and, as a function of τ with values in $C(\Omega)$, $y(\tau)$ is strongly measurable ⁽²⁾ on $(0, \infty)$. In particular, for fixed $\vec{\omega}$, $\hat{g}x(\tau, \vec{\omega}) \equiv y(\tau, \vec{\omega})$ is a numerical Lebesgue-measurable function of $\tau \in (0, \infty)$.

Definition. For $x(\cdot) \in B_1$, $\hat{A}x(\cdot) = z(\cdot)$, where, for $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, $z(\tau, \vec{\omega})$ is determined by the relations:

$$z(\tau, \omega) = \begin{cases} \frac{1}{\vec{\omega}\mathbf{n}} \int_0^\tau \exp\left(-\frac{\tau-\rho}{\vec{\omega}\mathbf{n}}\right) \hat{g}x(\rho, \vec{\omega}) d\rho, & \text{if } \vec{\omega}\mathbf{n} > 0; \\ \hat{g}x(\tau, \vec{\omega}), & \text{if } \vec{\omega}\mathbf{n} = 0; \\ \frac{1}{|\vec{\omega}\mathbf{n}|} \int_\tau^\infty \exp\left(-\frac{\rho-\tau}{|\vec{\omega}\mathbf{n}|}\right) \hat{g}x(\rho, \vec{\omega}) d\rho, & \text{if } \vec{\omega}\mathbf{n} < 0. \end{cases}$$

It turns out that, for $x(\cdot) \in B_1$, $z(\cdot) = \hat{A}x(\cdot)$ is an abstract function of $\tau \in (0, \infty)$ with values in $L_2(\Omega)$, depending continuously on τ . For $0 < \tau_1 < \infty$

$$\sup_{(\tau, \vec{\omega}) \in (0, \tau_1] \times \Omega} |z(\tau, \vec{\omega})| < \infty.$$

For fixed $\tau \in (0, \infty)$, $z(\tau, \vec{\omega})$ is continuous in the second argument at all points $\vec{\omega} \in \Omega$ for which $\vec{\omega}\mathbf{n} \neq 0$. If $\delta > 0$, $\tau_1 \in (0, \infty)$, then uniformly with respect to all those $\vec{\omega} \in \Omega$ for which $|\vec{\omega}\mathbf{n}| \geq \delta$,

$$\lim z(\tau, \vec{\omega}) =$$

$$= z(\tau_1, \vec{\omega}).$$

Put

$$z_0(\vec{\omega}) = \frac{1}{|\vec{\omega}\mathbf{n}|} \int_0^\infty \exp\left(-\frac{\rho}{|\vec{\omega}\mathbf{n}|}\right) \hat{g}x(0, \vec{\omega}) d\rho$$

for $\vec{\omega}\mathbf{n} < 0$, and $z_0(\vec{\omega}) = 0$ for $\vec{\omega}\mathbf{n} \geq 0$. Then, uniformly with respect to all $\vec{\omega} \in \Omega$, $|\vec{\omega}\mathbf{n}| \geq \delta$,

$$\lim_{\tau \rightarrow 0+0} z(\tau, \vec{\omega}) = z_0(\vec{\omega});$$

$$\sup_{\vec{\omega} \in \Omega} |z_0(\vec{\omega})| < \infty;$$

$z_0(\vec{\omega})$ is continuous at every point $\vec{\omega} \in \Omega$ for which $\vec{\omega}\mathbf{n} \neq 0$; $z_0 \in L_2(\Omega)$;

$$\lim_{\tau \rightarrow 0+0} \|z(\tau) - z_0\| = 0.$$

If $\tau_0 \in (0, \infty)$ is a point of strong continuity of $x(\cdot)$, then $z(\tau_0) \in C(\Omega)$, and uniformly with respect to all $\vec{\omega} \in \Omega$

$$\lim_{\tau \rightarrow \tau_0} z(\tau, \vec{\omega}) = z(\tau_0, \vec{\omega}).$$

If there exists the limit

$$\lim_{\tau \rightarrow 0+0} x(\tau) = x_0 \in L_2(\Omega),$$

then let $z_{00}(\vec{\omega}) = z_0(\vec{\omega})$ for $\vec{\omega}\vec{n} \neq 0$, and $= \hat{g}x_0(\vec{\omega})$ for $\vec{\omega}\vec{n} = 0$. Then $z_{00}(\vec{\omega})$ is continuous on the closed hemisphere

$$\Omega^- = \{\vec{\omega} \mid \vec{\omega} \in \Omega, \vec{\omega}\vec{n} \leq 0\},$$

and, uniformly with respect to $\vec{\omega} \in \Omega^-$,

$$\lim_{\tau=0+0} z(\tau, \vec{\omega}) = z_{00}(\vec{\omega}).$$

Theorem 1. For $\varepsilon \in (-1, 1)$, $\hat{A}(B_\varepsilon) \subset B_\varepsilon$; $\hat{A}(B_1) \subset B_1$. If $x(\cdot) \in B_\varepsilon$, $\varepsilon \in (-1, 1)$, then there exist constants $k_{3,4}$, $0 < k_{3,4} < \infty$, such that

$$\|\hat{A}^3 x(\tau)\| \leq k_3 e^{\varepsilon\tau}, \quad |\hat{A}^4 x(\tau, \vec{\omega})| \leq k_4 e^{\varepsilon\tau}$$

for all $\tau \in (0, \infty)$ and $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, respectively.

Let $Y(\xi) = 0$ for $\xi \leq 0$, $Y(\xi) = 1$ for $\xi > 0$;

$$1(\tau, \vec{\omega}) \equiv 1,$$

$$p_0(\tau, \vec{\omega}) = Y(\vec{\omega}\vec{n}) \exp(-\tau/\vec{\omega}\vec{n})$$

for $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$, and let F be an arbitrary bounded closed subset of the set $(0, \infty)$.

Theorem 2. Uniformly with respect to all $(\tau, \vec{\omega}) \in F \times \Omega$,

$$\lim_{n=\infty} g_0^{-n} \hat{A}^n 1(\tau, \vec{\omega}) = 0, \quad \sum_{\nu=0}^{\infty} g_0^{-\nu} \hat{A}^\nu p_0(\tau, \vec{\omega}) = 1.$$

Theorem 3. Let $\psi \in L_2(\Omega)$; $\psi \neq \vartheta$; let λ and s be complex numbers, $\lambda_1 \equiv \operatorname{Re} \lambda < 1$, $\lambda \in Z_0$, $|s| \geq g_0$, and

$$s(1 + \lambda \vec{\omega}\vec{n})\psi(\vec{\omega}) = \hat{g}\psi(\vec{\omega}), \quad \vec{\omega} \in \Omega.$$

This means that $\psi \in C(\Omega)$ and

$$\psi^+ = \sup_{\vec{\omega} \in \Omega} |\psi(\vec{\omega})| < \infty.$$

Put

$$x(\tau, \vec{\omega}) = e^{\lambda\tau} \psi(\vec{\omega}), \quad p(\tau, \vec{\omega}) = \psi(\vec{\omega}) p_0(\tau, \vec{\omega}).$$

Then $x(\cdot) \in B_\varepsilon$ for $\varepsilon > \lambda_1$, $p(\cdot) \in B_\varepsilon$ for $\varepsilon > -1$.

I. Uniformly with respect to $(\tau, \vec{\omega}) \in F \times \Omega$, there exists the limit

$$\bar{x}(\tau, \vec{\omega}) = \lim_{n=\infty} s^{-n} \hat{A}^n x(\tau, \vec{\omega}).$$

Moreover

$$\bar{x}(\tau, \vec{\omega}) = x(\tau, \vec{\omega}) - \sum_{\nu=0}^{\infty} s^{-\nu} \hat{A}^{\nu} p(\tau, \vec{\omega}).$$

The series on the right converges absolutely and uniformly with respect to $(\tau, \vec{\omega}) \in F \times \Omega$;

$$|\bar{x}(\tau, \vec{\omega}) - x(\tau, \vec{\omega})| \leq \psi^+$$

for all $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$.

II. $\bar{x}(\tau, \vec{\omega})$ is a continuous function of $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$; $\bar{x}(\cdot) \in B_{\varepsilon}$ for $\varepsilon > \lambda_1$.

III. If $\lambda_1 > 0$, then there exists $\tau_1 > 0$ such that $\bar{x}(\tau) \neq \vartheta$ for $\tau > \tau_1$. If $\lambda_1 \leq 0$, then $\bar{x}(\tau) \equiv \vartheta$ for $\tau \in (0, \infty)$.

IV. $\hat{A}\bar{x}(\tau) = s\bar{x}(\tau)$, $\tau \in (0, \infty)$.

Theorem 4. Let h_1 and h_2 be constants and, for $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$,

$$y(\tau, \vec{\omega}) = \tau + h_1 + h_2 \vec{\omega} \vec{n}.$$

Then $y(\cdot) \in B_{\varepsilon}$ for $\varepsilon > 0$.

I. Uniformly with respect to $(\tau, \vec{\omega}) \in F \times \Omega$, there exists the limit

$$\bar{y}(\tau, \vec{\omega}) = \lim_{n \rightarrow \infty} g_0^{-n} \hat{A}^n y(\tau, \vec{\omega});$$

$\bar{y}(\tau, \vec{\omega})$ does not depend on the choice of $h_{1,2}$ and

$$\begin{aligned} \bar{y}(\tau, \vec{\omega}) &= \tau + g_0 \frac{1 - \vec{\omega} \vec{n}}{g_0 - g_1} - \sum_{\nu=0}^{\infty} g_0^{-\nu} \hat{A}^{\nu} q(\tau, \vec{\omega}), \quad 2\partial e q(\tau, \vec{\omega}) = g_0(g_0 - g_1)^{-1} \times \\ &\quad \times (1 - \vec{\omega} \vec{n}) p_0(\tau, \vec{\omega}). \end{aligned}$$

The series on the right converges uniformly with respect to $(\tau, \vec{\omega}) \in F \times \Omega$, and for all $(\tau, \vec{\omega}) \in (0, \infty) \times \Omega$

$$0 \leq \sum_{\nu=0}^{\infty} g_0^{-\nu} \hat{A}^{\nu} q(\tau, \vec{\omega}) \leq g_0(g_0 - g_1)^{-1}.$$

II. $\bar{y}(\tau, \vec{\omega})$ is continuous in $(\tau, \vec{\omega})$ and nonnegative on $(0, \infty) \times \Omega$, $\bar{y}(\cdot) \in B_{\varepsilon}$ for $\varepsilon > 0$.

III. $\hat{A}\bar{y}(\tau) = g_0\bar{y}(\tau)$, $\tau > 0$.

Theorem 5. Let $g_0 < 1$, $\varepsilon \in (-\lambda_0, \lambda_0)$ (respectively $g_0 \leq 1$, $\varepsilon \in (-1, 0]$), $x(\cdot) \in B_{\varepsilon}$. Then:

I. The series

$$s(\tau, \vec{\omega}) = \sum_{n=0}^{\infty} \hat{A}^n x(\tau, \vec{\omega})$$

converges uniformly on $F \times \Omega$.

II. $s(\tau) = x_1(\tau) + u(\tau)$, $\tau > 0$, where $x_1(\cdot) \in B_\varepsilon$, $u(\tau, \vec{\omega})$ is continuous in $(\tau, \vec{\omega})$ on $(0, \infty) \times \Omega$;

$$\sup\{e^{-\varepsilon\tau}|u(\tau, \vec{\omega})| \mid (\tau, \vec{\omega}) \in (0, \infty) \times \Omega\} < \infty$$

(respectively, for every $\varepsilon' \in (\varepsilon, 0] \cap [-\lambda_0, 0]$,

$$\sup\{e^{-\varepsilon'\tau}|u(\tau, \vec{\omega})| \mid (\tau, \vec{\omega}) \in (0, \infty) \times \Omega\} < \infty).$$

III. $s(\cdot) \in B_{\varepsilon'}$ for $\varepsilon' > \varepsilon$ (respectively, for $\varepsilon' > \max\{\varepsilon, -\lambda_0\}$).

IV. $s(\tau) = \hat{A}s(\tau) + x(\tau)$, $\tau > 0$.

Let $k \in \mathfrak{R}$, $k > 0$ (1). Put in Theorem 3 $\lambda = k$, $s = 1$, $\psi = \psi_{kp}$, $x_{kp}(\tau) = e^{k\tau}\psi_{kp}$. Theorem 3 assigns to the function $x(\cdot) = x_{kp}(\cdot)$ the function $\bar{x}_{kp}(\cdot) = \bar{x}(\cdot)$. In the same way we define, for $g_0 = 1$,

$$\bar{y}_0(\cdot) = \bar{y}(\cdot),$$

where $\bar{y}(\cdot)$ is the function from Theorem 4. Let

$$X = \{\bar{x}_{kp}(\cdot) \mid k \in \mathfrak{R}, k > 0, p = 1, 2, \dots, p_k\}$$

and, for $g_0 = 1$,

$$X_1 = X \cup \{\bar{y}_0(\cdot)\}.$$

Then $X \subset \bar{B} = \bigcup_{\varepsilon < 1} B_\varepsilon$ and, for $g_0 = 1$, $X_1 \subset \bar{B}$. It is obvious that X (and for $g_0 = 1$, X_1) is a linearly independent system of functions. The only sign-constant function from X (for $g_0 = 1$, from X_1) is $\bar{x}_{\lambda_0 1}(\cdot)$ (respectively $\bar{y}_0(\cdot)$).

We shall call **problem A** problem (1) with

$$\mathcal{F}(\tau, \vec{\omega}) = B(\vec{\omega}) \exp(-\tau/\vec{\omega}\mathbf{n}),$$

where $B \in L_2(\Omega)$, $B(\vec{\omega}) = 0$ for $\vec{\omega}\mathbf{n} \leq 0$. Problem A is the problem of the diffusion of radiation through a semi-infinite layer, on whose boundary an externally incident flux is distributed according to the law $B(\vec{\omega})$. If $B = 0$, then problem A becomes homogeneous.

Theorem 6. In the class of functions $x(\cdot) \in \bar{B}$, each of which satisfies one of the conditions: a) $x(\cdot) \in B_\varepsilon$ for some $\varepsilon \in (-1, \lambda_0) \cup (-1, 0]$ or b)

$$\sup_{\tau > 0} \|x(\tau)\| < \infty,$$

there exists exactly one solution of problem A. This solution $\bar{s}(\tau)$ is represented by the Neumann series

$$\bar{s}(\tau, \vec{\omega}) = \sum_{n=0}^{\infty} \hat{A}^n \mathcal{F}(\tau, \vec{\omega}),$$

converging uniformly on $F \times \Omega$.

$$\sup_{\tau > 0} \|\bar{s}(\tau)\| < \infty; \quad \bar{s}(\cdot) \in B_\varepsilon,$$

for $\varepsilon' \in (-\lambda_0, 1)$.

Let $x(\cdot) \in B_\varepsilon$. Then for $\operatorname{Re} k > \varepsilon$ the function

$$\tilde{x}(k) = \int_0^\infty e^{-k\tau} x(\tau) d\tau$$

is defined and holomorphic, with values in $L_2(\Omega)$. The inversion formulas for this Laplace-Bochner integral depend on additional properties of $x(\cdot)$. Such properties are possessed by $x(\cdot)$ if it is a solution of problem A.

Theorem 7. Let $\varepsilon \in (-1, 1)$, $x(\cdot) \in B_\varepsilon$, and let $x(\cdot)$ be a solution of problem A. Then

$$\sup_{\tau > 0} e^{-\varepsilon\tau} \|x(\tau)\| < \infty;$$

there exists the limit

$$\lim_{\tau=0+0} x(\tau) =$$

$= x_0 \in L_2(\Omega)$, $x_0(\vec{\omega}) = B(\vec{\omega})$ for $\vec{\omega}\vec{n} > 0$; the Laplace transform $\tilde{x}(k)$, for $\operatorname{Re} k > \varepsilon$, satisfies the equation

$$(1 + k\vec{\omega}\vec{n})\tilde{x}(k) = g\tilde{x}(k) + (\vec{\omega}\vec{n})x_0.$$

Relying on (1), we can now analytically continue $\tilde{x}(k)$ to Z_0 . The singularities of $\tilde{x}(k)$ are the points $\Re \subset (-1, 1)$. It can be shown that for large $|\operatorname{Im} k|$,

$$\|\tilde{x}(k)\| \leq \operatorname{const} \cdot |\operatorname{Im} k|^{-1}.$$

This last circumstance makes it possible, using a suitable inversion formula for the Laplace integral, to obtain the following representations of $x(\tau)$. Let

$$s \in (-1, \min\{0, \varepsilon\}) \setminus \Re.$$

Then: a) for $g_0 < 1$ and for $g_0 = 1$, $\varepsilon < 0$,

$$\begin{aligned} x(\tau) &= - \sum_{\lambda \in \Re \cap (s, \varepsilon]} e^{\lambda\tau} \operatorname{sgn} \lambda \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p}) \psi_{\lambda p} + e^{s\tau} r_s(\tau) = \\ &= - \sum_{\lambda \in \Re \cap (0, \varepsilon]} \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p}) \bar{x}_{\lambda p}(\tau) + \bar{s}(\tau); \end{aligned}$$

b) for $g_0 = 1$, $\varepsilon \geq 0$,

$$\begin{aligned} x(\tau) &= -\frac{3}{4\pi}(1-g_1)(x_0, \vec{\omega}\vec{n})\tau + \frac{3}{4\pi}[(x_0, \vec{\omega}\vec{n})\vec{\omega}\vec{n} + (x_0, (\vec{\omega}\vec{n})^2)] - \\ &\quad - \sum_{\substack{\lambda \in \mathfrak{R} \cap (s, \varepsilon] \\ \lambda \neq 0}} e^{\lambda\tau} \operatorname{sgn} \lambda \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p}) \psi_{\lambda p} + e^{s\tau} r_s(\tau) = \\ &= - \sum_{\lambda \in \mathfrak{R} \cap (0, \varepsilon]} \sum_{p=1}^{p_\lambda} (x_0, (\vec{\omega}\vec{n})\psi_{\lambda p}) \bar{x}_{\lambda p}(\tau) - \frac{3}{4\pi}(1-g_1)(x_0, \vec{\omega}\vec{n})\bar{y}_0(\tau) + \bar{s}(\tau). \end{aligned}$$

In both cases

$$\sup_{\tau > 0} \|r_s(\tau)\| < \infty.$$

Theorem 8. Let $x(\cdot) \in \bar{B}$. $x(\cdot)$ is a solution of the homogeneous problem A if and only if $x(\cdot)$ is a linear combination of a finite number of functions from the system X (for $g_0 < 1$) or X_1 (for $g_0 = 1$). $x(\cdot)$ is a solution of the inhomogeneous problem A if and only if

$$x(\tau) = x_1(\tau) + \bar{s}(\tau),$$

where $x_1(\cdot) \in \bar{B}$, and $x_1(\cdot)$ is a solution of the homogeneous problem A.

In order that problem A admit nonnegative solutions, it is necessary and sufficient that

$$B(\vec{\omega}) \geq 0 \quad \text{for } \vec{\omega} \in \Omega.$$

If this condition is fulfilled, then Theorem 9 is true.

Theorem 9. Let $x_1(\cdot) \in \bar{B}$, and let $x_1(\cdot)$ be a nonnegative solution of problem A. Then

$$x_1(\tau) = x(\tau) + \bar{s}(\tau),$$

where $x(\cdot) \in \bar{B}$, and $x(\cdot)$ is a nonnegative solution of the homogeneous problem A. Moreover: a) if $g_0 < 1$, then (cf. (1))

$$x(\tau) = C\bar{x}_{\lambda_0 1}(\tau) = \frac{(x_0, (\vec{\omega}\vec{n})\Phi_{\lambda_0})}{(\vec{\omega}\vec{n}, \Phi_{\lambda_0}^2)} e^{\lambda_0 \tau} \Phi_{\lambda_0} - \frac{(x_0, (\vec{\omega}\vec{n})\Phi_{-\lambda_0})}{(\vec{\omega}\vec{n}, \Phi_{\lambda_0}^2)} e^{-\lambda_0 \tau} \Phi_{-\lambda_0} + e^{-\lambda_1 \tau} \rho(\tau);$$

b) if $g_0 = 1$, then

$$\begin{aligned} x(\tau) &= C\bar{y}_0(\tau) = -\frac{3}{4\pi}(1-g_1)(x_0, \vec{\omega}\vec{n})\tau + \\ &\quad + \frac{3}{4\pi}[(x_0, \vec{\omega}\vec{n})\vec{\omega}\vec{n} + (x_0, (\vec{\omega}\vec{n})^2)] + e^{-\lambda_1 \tau} \rho(\tau). \end{aligned}$$

In both cases

$$0 \leq C = \text{const} < \infty, \quad \sup_{\tau > 0} \|\rho(\tau)\| < \infty,$$

$$\lambda_1 = \min(\mathfrak{R}_0 \setminus \{\lambda_0\}) \quad (\text{cf. (1)}),$$

if $\mathfrak{R}_0$ contains more than one point, and λ_1 is an arbitrary number from $(\lambda_0, 1)$ otherwise.

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References Cited

1. M. V. Maslennikov, *DAN*, **168**, No. 3 (1966).
2. E. Hille, R. Phillips, *Functional Analysis and Semigroups*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.

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