

ON THE QUESTION OF ISOMORPHISMS OF AN ANALYTIC SPACE COMMUTING WITH A POWER OF THE DIFFERENTIATION OPERATOR

N. I. Nagnibida

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.50670>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88:513.83+517.948.35+517.948.5

MATHEMATICS

N. I. NAGNIBIDA

ON THE QUESTION OF ISOMORPHISMS OF AN ANALYTIC SPACE COMMUTING WITH A POWER OF THE DIFFERENTIATION OPERATOR

(Presented by Academician A. N. Kolmogorov on 26 VIII 1965)

The problem of finding the general form of isomorphisms Y of the space $\mathfrak{U}_\infty$ of entire functions that commute with the operator $D^n = d^n/dz^n$, $n \geq 1$, was posed by Delsarte and Lions in ⁽¹⁾. According to Theorem 2.1 of ⁽¹⁾, every such isomorphism Y satisfies, in the case $n \geq 2$, the relation

$$Y \exp(\lambda z) = \sum_{j=0}^{n-1} M_j(\lambda) \exp(\lambda z \omega^j), \quad \omega = \exp \frac{2\pi i}{n}, \quad (1)$$

where $M_j(\lambda)$, $0 \leq j \leq n-1$, are entire functions of exponential type with determinant

$$\Delta(\lambda) = \det \begin{vmatrix} M_0(\lambda) & M_1(\lambda) & \dots & M_{n-1}(\lambda) \\ M_{n-1}(\lambda\omega) & M_0(\lambda\omega) & \dots & M_{n-2}(\lambda\omega) \\ \dots & \dots & \dots & \dots \\ M_1(\lambda\omega^{n-1}) & M_2(\lambda\omega^{n-1}) & \dots & M_0(\lambda\omega^{n-1}) \end{vmatrix} \equiv \text{const} \neq 0. \quad (2)$$

Recently the result of Delsarte and Lions was carried over by I. Ya. Viner ⁽²⁾ to the spaces $\mathfrak{U}_R$, $0 < R < \infty$, of all single-valued analytic functions in the disk $|z| < R$ with the topology of compact convergence ⁽³⁾. It turns out, however, that the results of Delsarte–Lions and Viner are erroneous, i.e., the full group of isomorphisms of the space $\mathfrak{U}_R$ (respectively, $\mathfrak{U}_\infty$) commuting with D^n , $n \geq 2$, is not described by conditions (1), (2). Indeed, denoting by A_0 and A_1 the linear continuous operators in $\mathfrak{U}_R$ ($\mathfrak{U}_\infty$) defined by the relations

$$A_0 f(z) = \frac{f(z) + f(-z)}{2}, \quad A_1 f(z) = \frac{f(z) - f(-z)}{2},$$

let us consider the operators

$$Y = (I + E)A_0 + (D^3 + D^2 - D)A_1,$$

$$Y_1 = (D^2 - I)A_0 + (E + D - D^3)A_1;$$

where

$$If(z) = \int_0^z f(\zeta) d\zeta$$

and $Ef(z) = f(z)$. Since (as is easily verified on the basis elements z^k , $k = 0, 1, \dots$) $YD^2 = D^2Y$ and $YY_1 = Y_1Y = E$, it follows that Y is an isomorphism of the space $\mathfrak{U}_R$ ($\mathfrak{U}_\infty$) commuting with D^2 . Further, we have $Y1 = 1 + z$, although if the theorems of Delsarte-Lions and Viner were valid, then from (1), as $\lambda \rightarrow 0$, it would follow that $Y1 = \text{const} \neq 0$. Moreover, writing relation (1) for this operator, we easily obtain

$$M_0(\lambda) = \frac{1}{2} \left(\frac{1}{\lambda} + 1 - \lambda + \lambda^2 + \lambda^3 \right),$$

$$M_1(\lambda) = \frac{1}{2} \left(-\frac{1}{\lambda} + 1 - \lambda - \lambda^2 + \lambda^3 \right), \quad \Delta(\lambda) \equiv 1.$$

Thus, the operator Y constructed does not satisfy either of the conditions (1), (2).

The present paper is devoted to finding the general form of the isomorphisms of the spaces $\mathfrak{U}_R$ and $\mathfrak{U}_\infty$ considered here, relying on the matrix representation of linear continuous operators in these spaces ^(4,5).

1. If T is a linear operator in $\mathfrak{U}_R$ ($\mathfrak{U}_\infty$), commuting with D^n , $n \geq 1$, then the elements of its matrix $\{t_{i,k}\}_{i,k=0}^\infty$ in the power basis $\{z^k\}_{k=0}^\infty$, i.e.

$$Tz^k = \sum_{i=0}^{\infty} t_{i,k} z^i, \quad k = 0, 1, \dots,$$

are necessarily connected by the relations

$$t_{sn+p, mn+q} = \begin{cases} 0, & m < s, \\ \frac{(mn+q)! p!}{(sn+p)! [(m-s)n+q]!} t_{p, (m-s)n+q}, & 0 \leq s \leq m < \infty; \\ & 0 \leq p, q \leq n-1. \end{cases} \quad (3)$$

Let us introduce the operators A_q , $0 \leq q \leq n-1$, by putting

$$A_q f(z) = A_q \left(\sum_{i=0}^{\infty} a_i z^i \right) = \sum_{s=0}^{\infty} a_{sn+q} z^{sn+q},$$

where $f(z)$ is an arbitrary function from $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$).

From the relations (3) and the description of linear continuous operators in $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$)^(4,5) one may obtain

Theorem 1. In order that a linear operator T be a continuous operator in the space $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$), commuting with D^n , $n \geq 1$, it is necessary and sufficient that it have the form

$$T = \sum_{s=0}^{\infty} \sum_{q=0}^{n-1} \sum_{p=0}^{n-1} t_{p, sn+q} \frac{p!}{(sn+q)!} D^{sn+q-p} A_q^* \quad (4)$$

and satisfy the condition: for every $\rho < R$ ($\rho < +\infty$) there exists an $r = r(\rho) < R$ ($r < +\infty$) such that

$$\sup_{0 \leq mn+q < \infty} \sum_{s=0}^m \sum_{p=0}^{n-1} \frac{(mn+q)! p!}{(sn+q)! [(m-s)n+p]!} |t_{p, sn+q}| \frac{\rho^{(m-s)n+p}}{r^{mn+q}} < +\infty. \quad (5)$$

Theorem 2. For condition (5) to hold in the space $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$), it is necessary and sufficient that for each (some) ε , $\varepsilon > 0$, there exist a constant $M(\varepsilon) > 0$ such that

$$|t_{p,j}| \leq M(\varepsilon) \varepsilon^j, \quad j \geq 0; \quad 0 \leq p \leq n-1. \quad (6)$$

Corollary. In order that a linear operator T be a continuous operator in $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$), commuting with D^n , $n \geq 1$, it is necessary and sufficient that it have the form (4) and satisfy the corresponding condition (6).

Finally, note that when condition (6) is fulfilled, the functions

$$\psi_{p,q}(\lambda) = \sum_{m=0}^{\infty} t_{p, mn+q} \frac{\lambda^{mn}}{(mn+q)!}, \quad 0 \leq p, q \leq n-1,$$

are entire of class not exceeding $[1, 0]$ (respectively, functions of finite degree).

* In this formula, for $s = 0$ and $q < p$, $D^{q-p} = I^{p-q}$.

2. Let us proceed to find all isomorphisms T of the space $\mathfrak{U}_R$ ($\mathfrak{U}_{\infty}$) that commute with D^n , $n \geq 2$. Setting

$$\varphi_q(\lambda, z) = \sum_{m=0}^{\infty} \frac{\lambda^{mn} z^{mn+q}}{(mn+q)!}, \quad 0 \leq q \leq n-1,$$

we obtain

$$T\varphi_q(\lambda, z) = \sum_{p=0}^{n-1} p! \psi_{p,q}(\lambda) \varphi_p(\lambda, z), \quad 0 \leq q \leq n-1.$$

Proceeding from these relations and from the consequence of Theorem 2, it is easy to prove the following theorem.

Theorem 3. *In order that a linear operator T be an isomorphism of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$ that commutes with D^n , $n \geq 2$, it is necessary and sufficient that it have the form (4), satisfy the corresponding condition (6), and*

$$\det \|\psi_{p,q}(\lambda)\|_{p,q=0}^{n-1} \equiv \text{const} \neq 0.$$

Hence, in particular, it follows that every isomorphism of the space $\mathfrak{U}_R$ that commutes with D^n , $n \geq 2$, is at the same time an isomorphism of $\mathfrak{U}_\infty$.

3. Now let some isomorphism T of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$, commuting with D^n , $n \geq 2$, satisfy conditions (1), (2). Then from (1) and (4) it follows that

$$\sum_{j=0}^{n-1} M_j(\lambda) \lambda^p \omega^{pj} = \sum_{q=0}^{n-1} p! \lambda^q \psi_{p,q}(\lambda), \quad 0 \leq p \leq n-1.$$

Therefore, taking into account relations (3), we must conclude that the following theorem is true.

Theorem 4. *In order that an isomorphism of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$, commuting with D^n , $n \geq 2$, satisfy conditions (1), (2), it is necessary that its matrix have upper triangular form.*

4. Let us consider one application of the results presented above. Let

$$A = \sum_{j=0}^n a_j(z) D^j,$$

where $a_j(z) \in \mathfrak{U}_R(\mathfrak{U}_\infty)$ and $a_n(z) \equiv 1$. As is known ^(1,6,7), the operators A and D^n are equivalent to one another, i.e. there exists such an isomorphism T_0 of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$ (in this case there are infinitely many of them) that $T_0 D^n = A T_0$. In the works ^(1,6,7) it is indicated that the operator T_0 can be chosen so that

$$D^\nu T_0 f(z) \Big|_{z=0} = D^\nu f(z) \Big|_{z=0}, \quad 0 \leq \nu \leq n-1,$$

and all other transformation operators T_1 , $T_1 D^n = AT_1$, can be obtained by the formula $T_1 = T_0 T$, where T ranges over all possible isomorphisms of $\mathfrak{U}_R(\mathfrak{U}_\infty)$ that commute with D^n .

Let C be an operator in the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$, determined by the matrix $\{c_{i,k}\}_{i,k=0}^\infty$; $c_{i,k} = 0$, if $\min(i, k) \geq n$.

Theorem 5. *In order that there exist an isomorphism T_1 of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$ such that*

$$T_1 D^n = AT_1,$$

$$D^\nu T_1 f(z)|_{z=0} = D^\nu C f(z)|_{z=0}, \quad 0 \leq \nu \leq n-1; \quad n \geq 2, \quad (7)$$

it is necessary and sufficient that

$$\det \|c_{i,k}\|_{i,k=0}^{n-1} \neq 0. \quad (8)$$

Under these conditions the operator T_1 is determined uniquely.

Remark. If

$$B = \sum_{j=0}^n b_j(z) D^j, \quad b_j(z) \in \mathfrak{U}_R(\mathfrak{U}_\infty), \quad b_n(z) \equiv 1$$

and (8) is satisfied, then one can also assert the existence of such an isomorphism T_2 of the space $\mathfrak{U}_R(\mathfrak{U}_\infty)$ which satisfies relations (7) and $T_2 B = AT_2$.

In conclusion I express my sincere gratitude to K. M. Fishman for useful advice.

Chernivtsi State
University

Received
28 IV 1965

REFERENCES

1. J. Delsarte, J. L. Lions, *Comm. Math. Helv.*, **32**, No. 2, 113 (1957).
2. I. Ya. Viner, *UMN*, **20**, issue 1 (121), 185 (1965).
3. G. Köthe, *J. reine u. angew. Math.*, **191**, 30 (1953).
4. M. G. Khaplanov, *DAN*, **80**, No. 1, 21 (1951).

5. K. M. Fishman, *DAN*, **127**, No. 1, 40 (1959).
6. M. K. Fage, in: *Collected Papers: Studies on Modern Problems of the Theory of Functions of a Complex Variable*, Moscow, 1961, p. 468.
7. K. M. Fishman, *UMN*, **19**, issue 5 (119), 143 (1964).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.