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Abstract

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MATHEMATICS

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ON MEASURABLE POLYLINEAR AND POWER FUNCTIONALS IN CERTAIN LINEAR SPACES WITH MEASURE

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Let R be the real line, and let m be a countably additive normalized positive continuous ⁽¹⁾ measure, defined on a σ -algebra of subsets of R containing all Borel sets. Put

$$\Omega = \prod_{p \in P} R_p, \quad \mu = \prod_{p \in P} m_p,$$

where P is some abstract set. The set Ω is a linear space with respect to the usual operations. Elements of Ω will sometimes be called functions or sequences and denoted as follows: $\{x_p\}$, $\{x(p)\}$. By measurability with respect to some measure we shall mean measurability with respect to its Lebesgue extension.

In the present note we consider axiomatically definable measurable linear, polylinear, and power functionals in (Ω, μ) and in $(\Omega \times \dots \times \Omega, \mu \times \dots \times \mu)$, and also in (C, W) and in $(C \times \dots \times C, W \times \dots \times W)$, where C is the space of all continuous functions on the unit interval of the real axis, and W is Wiener measure in C . For the case when the space (Ω, μ) coincides with the sample space (Ω_g, μ_g) of an infinite sequence of independent Gaussian random variables with zero mathematical expectations and equal variances, measurable linear functionals in (Ω, μ) were axiomatically defined and studied in ⁽²⁾. Analogous results for the space (C, W) were obtained earlier by Cameron and Graves ^(5,6). Measurable quadratic functionals in (Ω_g, μ_g) , defined as limits of convergent double series, were considered in the work of Shilov and Fan Dyk Tinh ⁽³⁾. In the present note the results of ^(2,5,6) concerning measurable linear functionals are generalized, and certain analogous results are established for power and polylinear functionals. In particular, the connection is described between axiomatically definable measurable power functionals in (C, W) and multiple stochastic integrals considered in ^(4,7,8).

1. Measurable linear functionals

Let K be the set consisting of all functions from Ω , each of which differs from zero at no more than a finite number of points of P .

Theorem 1.1. If M is a measurable linear manifold of the space (Ω, μ) , having measure different from zero, and $M \cap K \neq \emptyset$, then $M \cap K = K$.

Theorem 1.2. The measure of every measurable linear manifold of the space (Ω, μ) is equal to zero or one.

Definition 1. A measurable linear functional (m.l.f.) in (Ω, μ) will mean any real function $f(\omega)$, $\omega \in \Omega$, possessing the following properties: 1) its domain of definition F is a measurable linear subspace of the space (Ω, μ) , with $\mu^*F = 1$; 2) $f(\omega)$ is linear on F ; 3) $f(\omega)$ is measurable with respect to the measure μ .

Theorem 1.3. Any two measurable linear functionals in (Ω, μ) either are distinct almost everywhere, or coincide almost everywhere; moreover, if all measures m_p are symmetric, the latter occurs if and only if these functionals coincide on the set K .

Theorem 1.4. If all measures $\{m_p\}_{p \in P}$ coincide with one and the same symmetric measure m , then every m.l.f. in (Ω, μ) satisfies the following conditions: 1) the set $\bar{P} = \{p : f(e_p) \neq 0, p \in P\}^*$ is at most countable; 2) $\sum_{p \in \bar{P}} (f(e_p))^2 < \infty$; 3) almost everywhere on (Ω, μ) the equality holds

$$f(\omega) = \sum_{p \in \bar{P}} f(e_p)x(p),$$

where the series on the right converges almost everywhere; 4) if

$$f(\omega) = \sum_{n=1}^{\infty} a_n x(p_n)$$

almost everywhere on (Ω, μ) , then $\{p_n : n = 1, 2, \dots\} = \bar{P}$ and $a_n = f(e_{p_n})$. If the measure m has finite variance, then the set of all measurable linear functionals in (Ω, μ) , supplied with the norm

$$|f| = \left[\int_{\Omega} f^2(\omega) \mu(d\omega) \right]^{1/2} = \left[\sum_{p \in \bar{P}} (f(e_p))^2 \right]^{1/2},$$

forms a complete pre-Hilbert space $H_l(\Omega, \mu)$.

2. Measurable polylinear functionals. Definition 2. By a measurable polylinear functional (m.p.f.) in (Ω, μ) we shall mean an m.l.f. in this space. Suppose that for $k = n - 1$ the definition of a measurable polylinear functional in $(\Omega^1 \times \dots \times \Omega^k, \mu^1 \times \dots \times \mu^k)$ has already been given. By an m.p.f. in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ we shall mean any real measurable function $b(\omega^1, \dots, \omega^n)$, defined almost everywhere on this space and satisfying the following conditions: 1) if $k_0 \in \{1, 2, \dots, n\}$ and for some ω^{k_0} from Ω^{k_0} the function

$$\varphi(\omega^1, \dots, \omega^{k_0-1}, \omega^{k_0+1}, \dots, \omega^n) = b(\omega^1, \dots, \omega^{k_0-1}, \omega^{k_0}, \omega^{k_0+1}, \dots, \omega^n)$$

is defined almost everywhere on

$$(\Omega^1 \times \dots \times \Omega^{k_0-1} \times \Omega^{k_0+1} \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^{k_0-1} \times \mu^{k_0+1} \times \dots \times \mu^n),$$

then it is an m.p.f. on this space; 2) if $k_0 \in \{1, 2, \dots, n\}$ and for some vector

$$(\omega^1, \dots, \omega^{k_0-1}, \omega^{k_0+1}, \dots, \omega^n)$$

from $(\Omega^1 \times \dots \times \Omega^{k_0-1} \times \Omega^{k_0+1} \times \dots \times \Omega^n)$ the function

$$\psi(\omega^{k_0}) = b(\omega^1, \dots, \omega^{k_0}, \dots, \omega^n)$$

is defined almost everywhere on $(\Omega^{k_0}, \mu^{k_0})$, then it is an m.l.f. on this space.

Theorem 2.1. Every m.p.f. in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ is defined on the set $K^1 \times \dots \times K^n$, where for each i , K^i is the set consisting of all possible functions from Ω^i , each of which differs from zero at no more than a finite number of points of P^i .

In all the theorems given below it is assumed that

$$(\Omega^1, \mu^1) = \dots = (\Omega^n, \mu^n) = (\Omega, \mu);$$

the set P is countable and all measures m_p coincide with one and the same symmetric measure m .

Theorem 2.2. Any two m.p.f. in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ either are distinct almost everywhere, or coincide almost everywhere. The latter occurs if and only if these functionals coincide on the set $K^1 \times \dots \times K^n$; moreover, if the measure m is Gaussian, the functionals under consideration also coincide on the set $l_2^1 \times \dots \times l_2^n$, where for each i , l_2^i is the set of all sequences from Ω^i with convergent sum of squares.

Theorem 2.3. Every m.p.f. $b(\omega^1, \dots, \omega^n)$ in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ has the following properties: 1) almost everywhere on $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ the equality holds

$$b(\{x^1(p^1)\}, \dots, \{x^n(p^n)\}) = \sum_{p^n \in P} \left(\dots \left(\sum_{p^1 \in P} b(e_{p^1}^1, \dots, e_{p^n}^n) x^1(p^1) \right) \dots \right) x^n(p^n),$$

where the repeated

$$* \quad e_{p_0} = \{x(p) : x(p) = 0, \text{ if } p \neq p_0, x(p_0) = 1\}.$$

the series on the right converges almost everywhere; 2)

$$\sum_{p^1, \dots, p^n \in P} (b(e_{p^1}^1, \dots, e_{p^n}^n))^2 < \infty;$$

3) if

$$b(\{x^1(p^1)\}, \dots, \{x^n(p^n)\}) = \sum_{p^n \in P} \left(\dots \left(\sum_{p^1 \in P} a_{p^1, \dots, p^n} x^1(p^1) \right) \dots \right) x^n(p^n)$$

almost everywhere on $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$, then

$$a_{p^1, \dots, p^n} = \bar{b}(e_{p^1}^1, \dots, e_{p^n}^n).$$

If the measure m has finite variance, then the set of all m.p.f.'s in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$, furnished with the norm

$$|b| = \left(\int_{\Omega^1 \times \dots \times \Omega^n} (b(\omega^1, \dots, \omega^n))^2 \mu^1(d\omega^1) \dots \mu^n(d\omega^n) \right)^{1/2} = \left(\sum_{p^1, \dots, p^n} (b(e_{p^1}^1, \dots, e_{p^n}^n))^2 \right)^{1/2},$$

forms a complete pre-Hilbert space $H_B^n(\Omega^i, \mu^i)$.

Definition 3. An m.p.f. $b(\omega^1, \dots, \omega^n)$ in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ will be called **symmetric** if, whatever $k_1, k_2 \in \{1, 2, \dots, n\}$ may be, almost everywhere in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$

$$b(\omega^1, \dots, \omega^{k_1}, \dots, \omega^{k_2}, \dots, \omega^n) = b(\omega^1, \dots, \omega^{k_2}, \dots, \omega^{k_1}, \dots, \omega^n).$$

Theorem 2.4. In every class of equivalence relative to the measure $\mu_g^1 \times \mu_g^2$ of symmetric m.p.f.'s in $(\Omega_g^1 \times \Omega_g^2, \mu_g^1 \times \mu_g^2)$ there is a functional $b(\omega^1, \omega^2)$ having the following properties:
1) its domain of definition coincides with the domain of convergence of the series

$$\sum_{k=1}^{\infty} \lambda_k \left(\sum_{n=1}^{\infty} u_{kn} x_n^1 \right) \times \left(\sum_{n=1}^{\infty} u_{kn} x_n^2 \right),$$

where $\|u_{nk}\|$ is some orthogonal matrix, $\{x_n^1\} = \omega^1$, $\{x_n^2\} = \omega^2$,

$$\sum_{k=1}^{\infty} \lambda_k^2 < \infty;$$

2) in this domain

$$b(\omega^1, \omega^2) = \sum_{k=1}^{\infty} \lambda_k \left(\sum_{n=1}^{\infty} u_{kn} x_n^1 \right) \times \left(\sum_{n=1}^{\infty} u_{kn} x_n^2 \right).$$

The domain of definition of the function $b(\omega^1, \omega^2)$ contains a certain measurable linear subspace of full measure of the space $(\Omega_g^1 \times \Omega_g^2, \dots)$ if and only if

$$\sum |\lambda_k| < \infty.$$

Theorem 2.5. Suppose the measure m has a finite moment of degree $2n$. Then, whatever the m.p.f. $b(\omega^1, \dots, \omega^n)$ in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ may be, there exists an m.p.f. $\tilde{b}(\omega^1, \dots, \omega^n)$, coinciding with it almost everywhere, for which the function $\tilde{b}(\omega, \dots, \omega)$ is defined almost everywhere in (Ω, μ) and is μ -measurable.

3. Measurable power functionals

Definition 4. A **measurable power functional** (m.p.w.f.) of degree n in (Ω, μ) will mean any real measurable function $s(\omega)$, defined almost everywhere on this space, satisfying the following conditions: 1) in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ there exists an m.p.f. $b(\omega^1, \dots, \omega^n)$ such that the domain of definition F of the function $b(\omega, \dots, \omega)$ coincides with the domain of definition of $s(\omega)$; 2) on F ,

$$s(\omega) = b(\omega, \dots, \omega).$$

Theorem 3.1. Every m.p.w.f. in (Ω, μ) is defined on all elements of the set K , and if the measure m is Gaussian, then also on all elements of the set l_2 .

Theorem 3.2. Suppose the measure m has a finite moment of degree $2n$. Then to every measurable multilinear functional $b(\omega^1, \dots, \omega^n)$ in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ there corresponds an m.p.w.f. of degree n , $s(\omega)$, in (Ω, μ) , for which on the set K the equality

$$b(\omega, \dots, \omega) = s(\omega)$$

holds.

Theorem 3.3. The values assumed by an m.p.w.f. $s(\omega)$ of degree n on the elements of the set K uniquely determine, up to equivalence with respect to the measure $\mu^1 \times \dots \times \mu^n$, a symmetric m.p.f. $b(\omega^1, \dots, \omega^n)$, for which almost everywhere on (Ω, μ)

$$s(\omega) = b(\underbrace{\omega, \dots, \omega}_n).$$

In the following three theorems it is assumed that the measure m has a finite fourth moment.

Theorem 3.4. In order that the difference of two second-degree m.s.f. coincide almost everywhere with a constant, it is necessary and sufficient that these functionals coincide on the set K .

Theorem 3.5. If $\{c_{p^1 p^2} : p^1, p^2 \in P; p^1 \neq p^2, c_{p^1 p^2} = 0\}$ and $\{a_{p^1 p^2} : p^1, p^2 \in P\}$ are sets of real numbers such that the series

$$\sum_{p^1, p^2 \in P} (a_{p^1 p^2} x(p^1) x(p^2) - c_{p^1 p^2})$$

converges in measure in the space (Ω, μ) under some order of summation, then in this space there exists a second-degree m.s.f. $s(\omega)$ such that, μ -almost everywhere,

$$s(\omega) = s(\{x(p)\}) = \sum_{p^1, p^2 \in P} (a_{p^1 p^2} x(p^1) x(p^2) - c_{p^1 p^2})$$

(with the same order of summation).

Theorem 3.6. Every second-degree m.s.f. $s(\omega) = b(\omega, \omega)$ in (Ω, μ) has the following properties: 1) under a suitable choice of constants c_p ($p \in P$), μ -almost everywhere the equality

$$s(\omega) \equiv s(\{x(p)\}) = \sum_{p^1 \neq p^2} b(e_{p^1}, e_{p^2}) x(p^1) x(p^2) + \sum_{p \in P} b(e_p, e_p) ((x(p))^2 - c_p)$$

holds independently of the order of summation of the series on the right-hand side (the first of these series converges in quadratic mean, the second almost everywhere); 2) if for the functional $s(\omega)$ almost everywhere in (Ω, μ) the equality

$$s(\omega) \equiv s(\{x(p)\}) = \sum_{p^1 \neq p^2} b_{p^1 p^2} x(p^1) x(p^2) + \sum_{p \in P} (b_{pp} (x(p))^2 - \tilde{c}_p)$$

also holds, where the series converge in measure μ under some order of summation, then

$$b_{p^1 p^2} = b(e_{p^1}, e_{p^2})^*$$

and, under the same order of summation,

$$\sum_{p \in P} (c_p - \tilde{c}_p) = 0.$$

4. Measurable polynomial and polylinear functionals in (C, W) and in $(C \times \dots \times C, W \times \dots \times W)$

The definitions given above retain their meaning for any linear space with measure, in particular for the space (C, W) . By virtue of the isomorphism of the spaces (Ω_g, μ_g) and (C, W) , all the properties established above for measurable functionals in (Ω, μ) and in $(\Omega^1 \times \dots \times \Omega^n, \mu^1 \times \dots \times \mu^n)$ can be reformulated for measurable functionals in (C, W) and in $(C \times \dots \times C, W \times \dots \times W)$.

Let $H_{B^n}^n(C, W)$ be the pre-Hilbert space of all i.p.f. in

$$(C \times \dots \times C, W \times \dots \times W),$$

where the product contains n factors, and let B^n be the unit n -dimensional cube.

Theorem 4.1. There exists a distance-preserving homomorphic mapping

$$\Phi : H_{B^n}^n(C, W) \rightarrow L_2(B^n),$$

having the following properties: 1) if $b(\dots) \in H_{B^n}^n(C, W)$, $f(t_1, \dots, t_n) = \Phi(b(\dots))$ ($f(\dots) \in L_2(B^n)$), then the multiple stochastic Itô-Wiener integral ^(4,7,8)

$$\int_0^1 \dots \int_0^1 f(t_1, \dots, t_n) dx(t_1) \dots dx(t_n)$$

almost everywhere in (C, W) coincides with a certain n -th degree m.s.f. in (C, W) , which takes on functions from C with integrable derivative the same values as $b(\omega, \dots, \omega)$ ($\omega \in C$); 2) the closure in $L_2(B^n)$ of the set $\Phi(H_{B^n}^n(C, W))$ coincides with the whole space $L_2(B^n)$.

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* Here it is assumed that the matrices $\|b_{nk}\|$ and $\|b(e_n, e_k)\|$ are symmetric.

Note: Figure translations are in progress. See original paper for figures.

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