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Abstract

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PHYSICS

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HOMOGENEOUS ANISOTROPIC COSMOLOGICAL MODEL WITH A MAGNETIC FIELD

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The article considers a homogeneous axisymmetric solution of the Einstein gravitational equations and Maxwell equations for the case when space is filled with a perfect substance and, in the accompanying frame, there is a magnetic field directed along the axis of symmetry. In this case the accompanying frame is synchronous, and the substance moves along geodesics. The presence of the magnetic field manifests itself in the geometry of space-time and substantially affects the character of the metric.

A cosmological model with a magnetic field was considered by Ya. B. Zel'dovich⁽²⁾, and subsequently by A. G. Doroshkevich⁽³⁾. In the present article the case of a closed and an open model for dust-like matter is considered in greater detail. The notation of the book⁽¹⁾ will be used.

1. As space-time coordinates we choose an accompanying coordinate system in which the substance is at rest. It is assumed that in this system the metric has axial symmetry and that the distribution of matter is homogeneous (i.e., depends only on the proper time τ). It is also assumed that the accompanying system used is synchronous. Then, denoting the axis of symmetry by z , and the radial and angular coordinates by χ and φ , we obtain the expression for the metric in the form

$$-ds^2 = -c^2 d\tau^2 + a^2(\tau) [d\chi^2 + f^2(\chi) d\varphi^2] + b^2(\tau) dz^2. \quad (1)$$

The coordinates and time are numbered by means of $x^0 = c\tau$, $x^{1,2,3} = \chi, \varphi, z$. Latin indices run over the values from 0 to 3.

Einstein's gravitational equations:

$$R_i^k - \frac{1}{2} \delta_i^k R = (8\pi k/c^4) T_i^k. \quad (2)$$

The energy-momentum tensor on the right-hand side of (2) is the sum of two parts: the energy-momentum tensor of a perfect gas and the energy-momentum tensor of the electromagnetic field. Since an accompanying system is used, among the components of the energy-momentum tensor of the perfect gas only the diagonal components are different from 0: $T_1^1 = T_2^2 = T_3^3 = p$, p is the pressure, $T_0^0 = -e$, e is the density of internal energy. It is assumed that among the components of the electromagnetic-field tensor F_{ik} in the accompanying system used only $F_{12} = -F_{21} = F$ differ from 0. Then, for the energy-momentum tensor of the electromagnetic field, only diagonal components differ from 0, equal to $T_1^1 = T_2^2 = -T_3^3 = -T_0^0 = W = F^2/8\pi a^4(\tau)f^2(\chi)$.

Einstein's equations (2) in the case under consideration have the form

$$R_0^0 - \frac{R}{2} : \quad -\frac{\dot{a}^2}{c^2 a^2} - 2\frac{\dot{a}\dot{b}}{c^2 ab} + \frac{1}{a^2} \frac{f''(\chi)}{f(\chi)} = -\frac{8\pi k}{c^4} (e + W), \quad (3)$$

$$R_1^1 - \frac{R}{2} = R_2^2 - \frac{R}{2} : \quad -\frac{\ddot{a}}{c^2 a} - \frac{\dot{a}\dot{b}}{c^2 ab} - \frac{\ddot{b}}{c^2 b} = \frac{8\pi k}{c^4} (p + W), \quad (4)$$

$$R_3^3 - \frac{R}{2} : \quad -\frac{2\ddot{a}}{c^2 a} - \frac{\dot{a}^2}{c^2 a^2} + \frac{1}{a^2} \frac{f''(\chi)}{f(\chi)} = \frac{8\pi k}{c^4} (p - W). \quad (5)$$

A dot denotes differentiation with respect to τ , and a prime denotes differentiation with respect to χ .

To the gravitational equations one must add Maxwell's equations

$$\partial F_{ik}/\partial x^l + \partial F_{kl}/\partial x^i + \partial F_{li}/\partial x^k = 0. \quad (6)$$

Since, among the components F_{ik} , only $F_{12} = -F_{21}$ differ from 0, and they may depend on x^0 and x^1 , the only component of (6) that does not vanish identically is that with $l = 0$, $i, k = 1, 2$. It gives $\partial F_{12}/\partial x^0 = 0$, i.e. $F = F(\chi)$.

It is convenient to use the equations $T_{i;k}^k = 0$ contained in (2). The components with $i = 0$ and $i = 1$ do not vanish identically. For $i = 0$ we have

$$(2\dot{a}/a) + (\dot{b}/b) = -\dot{e}(e + p). \quad (7)$$

For $i = 1$ we have (taking into account that p and e depend on τ) $2f'(\chi)/f(\chi) = (F^2)' / F^2$, i.e.

$$F^2(\chi) = f^2(\chi) K_1^2, \quad K_1^2 = \text{const}. \quad (8)$$

The energy density of the electromagnetic field is then equal to $W = K_1^2/8\pi a^4(\tau)$, in accordance with the homogeneity condition.*

From (3) and (5), since a , b , p , e , and W depend on τ , it follows that

$$f''(\chi)/f(\chi) = \mp \alpha^2, \quad \alpha^2 = \text{const.} \quad (9)$$

If $\alpha = 0$, then $f(\chi) = \text{const} \cdot \chi$ (in view of χ as a radial coordinate one must have $f(0) = 0$). By changing the scale of φ , the constant can be made equal to 1, and we obtain $f(\chi) = \chi$. This case corresponds to the flat model $(^2, ^3)$.

For $\alpha \neq 0$, changing the scale of χ , one may set $\alpha = 1$. Taking the upper sign in (9), we obtain $f(\chi) = c_1 \sin \chi + c_2 \cos \chi$. The constant c_2 must be set equal to 0 so that $f(0) = 0$, and the constant c_1 , by changing the scale of φ , can be made equal to 1. We then obtain $f(\chi) = \sin \chi$. This case corresponds to the closed model $(^2, ^3)$. The coordinate χ varies from 0 to π .

Taking the lower sign in (9), we obtain $f(\chi) = \text{sh } \chi$. This case corresponds to the open model $(^2, ^3)$. The coordinate χ may then vary from 0 to ∞ .

The three-dimensional scalar curvature P of the 3-space $x^{1,2,3}$ is equal to $P = -2(f''/f)/a^2 = \pm 2/a^2(\tau)$. At each instant τ in the closed model the 3-space is a space of constant positive curvature (with infinite volume, since z varies over infinite limits), while in the open model it is a space of constant negative curvature.

Equation (3), using the “Hubble constants” $h_\perp = \dot{a}/a$ and $h_\parallel = \dot{b}/b$, is written in the form

$$h_\perp(h_\perp + 2h_\parallel) - 8\pi k(e + W)/c^2 = \mp \alpha^2(c^2/a^2),$$

which can serve as a criterion for the type of model $(^2, ^3)$.

2. In what follows we shall consider the case of dust matter ($p = 0$). From (7) it then follows that

$$e = K/a^2 b, \quad K = \text{const.} \quad (10)$$

For $p = 0$, $e = \mu c^2$, where μ is the rest-mass density; (10) is also the continuity integral for mass and expresses conservation of mass in a “fluid” volume.

From the gravitational equations (3)–(5) we shall use equations (3) and (5), since equation (7) has already been used. Taking (8), (9), and (10) into account, we have

$$2a\ddot{a} + \dot{a}^2 + c^2(\pm \alpha^2) - B_1^2(c^2/a^2) = 0; \quad (11)$$

$$\frac{\dot{a}^2}{c^2 a^2} + 2 \frac{\dot{a}\dot{b}}{c^2 ab} + \frac{(\pm \alpha^2)}{a^2} - \frac{2D}{a^2 b} - \frac{B_1^2}{a^4} = 0. \quad (12)$$

* The strength of the magnetic field directed along the z -axis is $H = K_1/a^2(\tau)$.

In equations (11)–(12) $B_1^2 = kK_1^2/c^4 = \text{const}$, $2D = 8\pi kK/c^4 = \text{const}$.

Equation (11) determines $a(\tau)$. Integrating once, we obtain

$$\dot{a}^2/c^2 = \mp\alpha^2 - (B_1^2/a^2) + (2A/a), \quad 2A = \text{const}. \quad (13)$$

It is convenient, instead of τ , to introduce a variable $\eta = \eta(\tau)$ according to

$$c d\tau = a d\eta. \quad (14)$$

Equation (13) is now written in the form

$$(da/d\eta)^2 = \mp\alpha^2 a^2 + 2Aa - B_1^2. \quad (15)$$

Formulas (15) and (14) determine $a(\eta)$ and $\tau(\eta)$. In the case of the **closed** model (in (15) one must take the upper sign and $\alpha = 1$), integrating, we obtain (we shall assume a positive; then the constant A must be > 0)

$$a(\eta) = A [1 - (1 - B^2)^{1/2} \cos \eta], \quad c\tau(\eta) = A [\eta - (1 - B^2)^{1/2} \sin \eta],$$

$$B^2 = B_1^2/A^2 = \text{const}. \quad (16)$$

The additive constants in η and τ have been chosen so that at $\eta = 0$, $\tau = 0$, and a is minimal.

Equations (16), for $B \neq 0$ (in the presence of a magnetic field), determine a trochoid in parametric form. The “radius of curvature” a pulsates, varying within the limits from $A - (A^2 - B_1^2)^{1/2}$ to $A + (A^2 - B_1^2)^{1/2}$, and nowhere becomes zero. At the same time, for the values $a_{\min}$ and $a_{\max}$ the derivative $da/d\tau$ is equal to 0. In the absence of a magnetic field ($B = 0$), equations (16) determine a cycloid. In this case the minimum value of a is equal to 0, and the energy density e , according to (10), then becomes ∞ . An axisymmetric homogeneous cosmological model in the absence of a magnetic field was considered in (4).

In the **open** model, from (15) and (14) we obtain (we assume a positive; the constant A may have any sign and may be equal to 0):

$$a(\eta) = -A + (A^2 + B_1^2)^{1/2} \text{ch } \eta, \quad c\tau(\eta) = -A\eta + (A^2 + B_1^2)^{1/2} \text{sh } \eta. \quad (17)$$

The quantity a varies from the minimum value $a_0 = -A + (A^2 + B_1^2)^{1/2}$ to ∞ , first decreasing to a_0 , and then increasing to ∞ ; moreover, in the presence of a magnetic field ($B_1 \neq 0$) a_0 does not become 0. In the absence of a magnetic field ($B_1 = 0$) and for $A > 0$, a_0 becomes 0.

In the **flat** model ($\alpha = 0$) we have

$$a(\eta) = A(B^2 + \eta^2)/2, \quad c\tau(\eta) = A[(B^2/2)\eta + (\eta^3/6)]. \quad (18)$$

Again, for $B \neq 0$, the minimum value a/A does not become 0.

The function $b(\tau)$ in metric (1) is found from equation (12). It is convenient to determine b as a function of η . Taking (13) into account, from (12) we obtain

$$(db/d\eta)(da/d\eta) = [(B_1^2/a) - A]b + Da. \quad (19)$$

In the **closed** model, using (16), we obtain

$$\frac{b(\eta)}{D} = \frac{A}{a(\eta)} \left[2 - \frac{2 - B^2}{(1 - B^2)^{1/2}} \cos \eta - (1 - B^2)^{1/2}(\eta - E) \sin \eta \right], \quad E = \text{const.} \quad (20)$$

The constant D plays the role of a scale factor of the coordinate z (just as A plays the role of a scale factor of the coordinates χ and τ). In fact, the essential constants prove to be B^2 and E .

From (20), owing to the presence of the product of the oscillating factors $\sin \eta$ and η , it follows that $b(\eta)$ vanishes, and does so an infinite number of times. The distances along the z -axis between “fluid” particles successively decrease to zero (“simultaneously” for the whole space, turning into 0) and increase.

In the **open** model we obtain

$$\frac{b(\eta)}{D} = \frac{1}{a(\eta)} \left[2A - \frac{2A^2 + B_1^2}{(A^2 + B_1^2)^{1/2}} \text{ch } \eta + (A^2 + B_1^2)^{1/2}(\eta - E) \text{sh } \eta \right], \quad E = \text{const.} \quad (21)$$

The expression in square brackets in (21) has opposite signs at $\eta = 0$ and as $\eta \rightarrow \pm\infty$; the function $b(\eta)$ vanishes twice: for $\eta < 0$ and for $\eta > 0$ (when $B_1 \neq 0$).

Finally, in the **flat** model we have

$$b(\eta)/D = [A/a(\eta)] \left[-(B^4/4) + (B^2/2)\eta^2 + \frac{1}{12}\eta^4 + E\eta \right], \quad E = \text{const.} \quad (22)$$

$b(\eta)$ vanishes twice: for $\eta < 0$ and for $\eta > 0$ ($B \neq 0$). The unused Maxwell equations $(-g)^{-1/2} \partial[(-g)^{1/2} F^{ik}] / \partial x^k = (4\pi/c) j^i$ determine the 4-current j^i . Computing the left-hand side, we obtain, taking (8) into account, that $j^i = 0$. The equality of the 4-current to zero means the absence of a ponderomotive force. This is consistent with the fact that, in the present solution, the comoving system is simultaneously synchronous and the matter moves along geodesics. In a synchronous system, as is known, the vanishing of the quantity $[\partial \ln(-g) / \partial t]^{-1}$ at some instant τ is inevitable^(1,5). In the solution considered, when $b(\tau) = 0$ the determinant $-g = a^4 b^2 f^2$ vanishes; at the same time $e = -T_i^i = (c^4/8\pi k) R$ tends to ∞ , so that the singularity has a physical character.

Let us note that a solution analogous to the one considered also exists in the absence of matter. The role of the comoving system is then played by the system in which the energy-momentum tensor of the electromagnetic field is diagonal. Formulas (16)–(18) for a and t are preserved. The dependence $b(\eta)$ is now determined from equation (19) with $D = 0$, whose solution has the form $b(\eta) = \text{const} \cdot \sin \eta/a(\eta)$ in the closed model, $b(\eta) = \text{const} \cdot \text{sh } \eta/a(\eta)$ in the open model, and $b(\eta) = \text{const} \cdot \eta/a(\eta)$ in the flat one. The quantity $b(\eta)$ vanishes in the first case periodically, in the second at $\eta = 0$, and in the third at $\eta = 0$ and $\eta = \infty$.

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Note: Figure translations are in progress. See original paper for figures.

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