

HYPOELLIPTIC EQUATIONS IN CONVOLUTIONS

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Abstract

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MATHEMATICS

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HYPOELLIPTIC EQUATIONS IN CONVOLUTIONS

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1. In recent years a number of profound works have appeared on the theory of singular integro-differential (pseudodifferential) operators (¹⁻⁸). It has been found that these operators possess many important properties inherent in differential operators; in particular, the "principle of locality" holds for them. This principle makes it possible to construct for pseudodifferential operators not only a theory of boundary-value problems, but also a local theory.

In the present paper the local properties of pseudodifferential operators are studied. We single out a broad class of operators with nonhomogeneous symbol, which is a natural generalization of differential operators of constant strength (⁹). By the methods of (⁸), for this class of operators a calculus is constructed, analogous to the calculus of homogeneous operators. Next, hypoelliptic operators are introduced, generalizing formally hypoelliptic differential operators (^{9,10}). An essential role throughout the exposition is played by the spaces H^μ of tempered distributions; the theory of these spaces was developed in (^{9,11}).

2. Definitions. $x = (x^1, \dots, x^n)$ are points of the n -dimensional real space R^n , $\xi = (\xi_1, \dots, \xi_n)$ are variables dual with respect to the form $x \cdot \xi = x^1 \xi_1 + \dots + x^n \xi_n$; $D_k = -i \partial / \partial x^k$, $D = (D_1, \dots, D_n)$; $\partial_k = \partial / \partial \xi_k$, $\partial = (\partial_1, \dots, \partial_n)$. If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a set of n integers (a multi-index), then $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$. The quantities D^α , x^α , ∂^α are defined similarly. C^∞ , $C^\infty(\Omega)$ are the spaces of infinitely differentiable functions in R^n (in the domain Ω); C_0^∞ ($C_0^\infty(\Omega)$) are the functions from C^∞ ($C^\infty(\Omega)$) with compact supports. $\mathfrak{S}$ is the space of functions $u(x) \in C^\infty$ for which $\sup |x^\alpha D^\beta u(x)| < \infty$ for all multi-indices α, β ; $\mathfrak{S}'$ is the conjugate space of tempered distributions. If $u \in \mathfrak{S}'$, then by $\tilde{u}$ we shall denote the Fourier transform of the distribution u , and by $H^{(s)}$ the set of those $u \in \mathfrak{S}'$ for which $\tilde{u}$ is a locally integrable function and $(1 + |\xi|)^s \tilde{u}(\xi) \in L_2$. By $H^{(\infty)}$, $H^{(-\infty)}$ we denote the intersection (union) of all spaces $H^{(s)}$, $-\infty < s < \infty$, endowed with the natural topology of the projective (inductive) limit.

3. Definition of pseudodifferential operators. Pseudolocality. We shall call symbols the functions $a(x, \xi)$, defined for all pairs $(x, \xi) \in R^n \times R_n$ and

having a limit as $|x| \rightarrow \infty$, i.e. $a(x, \xi) = a(\xi) + a'(x, \xi)$, where $a'(x, \xi) \rightarrow 0$ as $|x| \rightarrow \infty$. To each symbol, following ⁽⁸⁾, one can associate operators

$$\mathcal{A}u(x) = (2\pi)^{-n/2} \int e^{ix \cdot \eta} a(x, \eta) \tilde{u}(\eta) d\eta, \quad (1)$$

$$Au(x) = (2\pi)^{-n/2} \int e^{ix \cdot \xi} \left[(2\pi)^{-n/2} \int e^{-iy \cdot \xi} a(y, \xi) u(y) dy \right] d\xi. \quad (2)$$

The operators (1), (2) are initially defined in $\mathfrak{S}$ or $H^{(\infty)}$. Using the formal adjointness of the operators $\mathcal{A}$ and $\bar{A}$ ($\bar{A}$ is the operator

(2)', corresponding to the symbol $\bar{a}(x, \xi)$, i.e.

$$(\mathcal{A}u, v) = (u, \bar{A}v); \quad u, v \in \mathcal{S}, \quad (3)$$

they can be extended to the dual spaces $\mathcal{S}'$, $H^{(-\infty)}$.

By the methods of ⁽⁸⁾ one proves

Proposition 1. If $a(x, \xi) \in C^\infty$ with respect to x ,

$$|a(\xi)| < c(1 + |\xi|)^N, \quad (4)$$

and for all multiindices α

$$\int |D^\alpha a'(x, \xi)| dx \leq c_\alpha (1 + |\xi|)^{N + \theta|\alpha|}, \quad \theta < 1, \quad (4')$$

then the operators (1), (2) are continuous as operators acting from $H^{(\infty)}$ to $H^{(\infty)}$ and from $H^{(-\infty)}$ to $H^{(-\infty)}$. If, moreover, $a(x, \xi) \in C^\infty$ with respect to x and ξ , and the derivatives $D^\beta a(x, \xi)$ satisfy conditions (4), (4') (the numbers N and θ may depend on β), then the operators (1), (2) are continuous as operators acting from $\mathcal{S}$ to $\mathcal{S}$ and from $\mathcal{S}'$ to $\mathcal{S}'$.

An operator $\mathfrak{A} : \mathcal{S}' \rightarrow \mathcal{S}'$ is called **pseudolocal** if $\{u \in \mathcal{S}', u = 0 \text{ in } \Omega \Rightarrow \mathfrak{A}u \in C^\infty(\Omega)\}$, where Ω is some domain in R^n . As noted in ⁽⁸⁾, if the operator $\mathfrak{A}$ is pseudolocal and $\mathfrak{A}\mathcal{S} \subset \mathcal{S}$, then $\{u \in \mathcal{S}', u \in C^\infty(\Omega) \Rightarrow \mathfrak{A}u \in C^\infty(\Omega)\}$. Modifying the arguments of ⁽³⁾, one can prove

Proposition 2. Let the continuity conditions for the operators (1), (2) in $\mathcal{S}'$ (Proposition 1) be fulfilled, and suppose that for any $s > 0$ there exists $N = N(s)$ such that for all β

$$\sum_{|\alpha|=N} \left[\delta^\alpha a(\xi) + \int |D^\beta \delta^\alpha a'(x, \xi)| dx \right] \leq c_\beta (1 + |\xi|)^{-s}.$$

Then the operators (1), (2) are pseudolocal.

4. Pseudodifferential operators in the spaces H^μ . We shall denote all functions of the form $\text{const} \cdot (1 + |\xi|)^N$ occurring below by $\rho(\xi)$. As in ⁽¹¹⁾, let $\mathfrak{B}$ denote the class of nonnegative weight functions satisfying the conditions: for all $\xi, \eta \in R_n$,

$$\mu(\xi)\mu^{-1}(\eta) \leq \rho(\xi - \eta). \quad (5)$$

By H^μ we denote the set of such functions $u \in H^{(-\infty)}$ that $\mu(\xi)\tilde{u}(\xi) \in L_2$. In view of (5), H^μ will be a module over $\mathcal{S}$. By $H_{\text{loc}}^\mu(\Omega)$ we denote the set of such distributions $u \in \mathcal{S}'$ for which $\varphi u \in H^\mu$ for all $\varphi \in C_0^\infty(\Omega)$.

Put

$$\mu_{(s)}(\xi) = (1 + |\xi|)^s \mu(\xi). \quad (6)$$

We shall say that μ is the order of the operator $\mathfrak{A} : H^{(\infty)} \rightarrow H^{(\infty)}$ equal to r , if $\|\mathfrak{A}u\|_{\nu_{(s)}} \leq C\|u\|_{\mu\nu_{(s+r)}}$ for arbitrary $\nu \in \mathfrak{B}$ and $-\infty < s < \infty$. If $r = 0$ ($r < 0$), then the operator $\mathfrak{A}$ is called μ -bounded (μ -smoothing). If $\mu \equiv \text{const}$, then we shall simply call the operator bounded (smoothing).

Proposition 3. Let $\mu(\xi) \in \mathfrak{B}$ and for all α

$$|a(\xi)| < c\mu(\xi), \quad \int |D^\alpha a'(x, \xi)| dx < c_\alpha \mu(\xi), \quad (7)$$

Then the operators (1), (2) are μ -bounded.

5. Regular pseudodifferential operators. Let $\mathfrak{B}_0$ denote the class of weights $\mu \in \mathfrak{B}$ satisfying the additional condition

$$|\mu(\xi) - \mu(\eta)| < \rho(\xi - \eta)(1 + |\eta|)^{-\sigma} \mu(\eta), \quad \sigma > 0, \quad (8)$$

and by $\mathfrak{S}_0^\mu$ the class of symbols $a(x, \xi)$ for which inequalities (7) are satisfied and, in addition,

$$|a(\xi) - a(\eta)| < \rho(\xi - \eta)(1 + |\eta|)^{-\sigma} \mu(\eta), \quad (9)$$

$$\int |D_a^\alpha a'(x, \xi) - D_a^\alpha a'(x, \eta)| dx < c_\alpha \rho(\xi - \eta)(1 + |\eta|)^{-\sigma} \mu(\eta). \quad (9')$$

Symbols $a(x, \xi) \in \mathfrak{S}_0^\mu$ will be called μ -regular.

Proposition 4. If $a(x, \xi) \in \mathfrak{S}_0^\mu$, then $\mathcal{A} - A$ is a μ -smoothing operator.

Proposition 5. If $a(x, \xi) \in \mathfrak{S}_0^\mu$, $b(x, \xi) \in \mathfrak{S}_0^\nu$, then

$$c(x, \xi) = a(x, \xi)b(x, \xi) \in \mathfrak{S}_0^{\mu\nu},$$

and the operator $\mathcal{A}\mathcal{B} - \mathcal{C}$ is μ -smoothing. (Here $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are the operators (1) corresponding to the symbols a, b, c .)

Corollary. If $a(x, \xi) \in \mathfrak{S}_0^\mu$ and $\varphi(x) \in \mathcal{S}$, then the operator $[\mathcal{A}, \varphi]$ is μ -smoothing.

For μ -regular symbols one can refine Proposition 3.

Proposition 6. Let $a(x, \xi) \in \mathfrak{S}_0^\mu$, and suppose that $|a(x, \xi)| \leq \mu(\xi)$. Then for every $\varepsilon > 0$ and arbitrary $\nu \in \mathfrak{B}$, $-\infty < s < \infty$,

$$\|\mathcal{A}u\|_{\nu(s)} \leq (1 + \varepsilon)\|u\|_{\mu\nu(s)} + C(\mu, \nu, s, \varepsilon)\|u\|_{\mu\nu(s-1)}. \quad (10)$$

The proofs of Propositions 3–6 require no new ideas compared with (8).

6. Hypoelliptic pseudodifferential operators

Theorem 1 (on global regularity). Let $a(x, \xi) \in \mathfrak{S}_0^\mu$, and suppose that for sufficiently large ξ the lower bound

$$|a(x, \xi)| > c\mu(\xi), \quad |\xi| > R \quad (11)$$

holds. Then

$$\{u \in H^{(-\infty)}, \mathcal{A}u \in H^{\nu(s)} \Rightarrow u \in H^{\mu\nu(s)}\}.$$

Proof. Condition (11) makes it possible to construct for the operator (1) a left regularizer, i.e., a $1/\mu$ -bounded operator $\mathfrak{R}$ such that

$$\mathfrak{R}\mathcal{A} = E + T,$$

where T is a smoothing operator (for some $\sigma > 0$). As $\mathfrak{R}$ one may take the operator (1) corresponding to the symbol

$$\varphi(\xi)/a(x, \xi),$$

where $\varphi(\xi) \in C^\infty$, $\varphi(\xi) = 0$ for $|\xi| \leq R$ and $\varphi(\xi) = 1$ for $|\xi| \geq R + 1$. From the inclusion $u \in H^{(-\infty)}$ it follows that $u \in H^{\mu\nu(s_0)}$. Since $\mathcal{A}u \in H^{\nu(s)}$, we have

$$u = \mathfrak{R}\mathcal{A}u - Tu \in H^{\mu\nu(s_1)},$$

where $s_1 = \min(s, s_0 + \sigma)$. Iterating this argument, we prove the theorem.

Theorem 2 (on local regularity). Let the symbol $a(x, \xi)$ satisfy the conditions of Theorem 1 and Proposition 2. Then

$$\{u \in \mathcal{S}', \mathcal{A}u \in H_{\text{loc}}^{\nu(s)}(\Omega) \Rightarrow u \in H_{\text{loc}}^{\mu\nu(s)}(\Omega)\},$$

where Ω is a bounded domain in R^n .

Corollary.

$$\{u \in \mathcal{S}', \mathcal{A}u \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)\},$$

i.e., a pseudodifferential operator satisfying the conditions of Theorem 2 is hypoelliptic.

Proof. If the domain Ω is bounded, then there exists such an s_0 that

$$u \in H_{\text{loc}}^{\mu\nu(s_0)}(\Omega).$$

Let $\varphi, \psi \in C_0^\infty(\Omega)$, with $\psi(x) = 1$ for $x \in \text{supp } \varphi$. Then

$$\mathcal{A}(\varphi u) = \varphi \mathcal{A}u + [\mathcal{A}, \varphi]u = \varphi \mathcal{A}u + [\mathcal{A}, \varphi](\psi u) + \varphi \mathcal{A}((1 - \psi)u). \quad (12)$$

By assumption, $\varphi \mathcal{A}u \in H^{\nu(s)}$; by Proposition 5,

$$[\mathcal{A}, \varphi](\psi u) \in H^{\nu(s_0 + \sigma)};$$

and by Proposition 2 (pseudolocality),

$$\varphi \mathcal{A}((1 - \psi)u) \in C_0^\infty(\Omega).$$

Thus the right-hand side of (12) belongs to $H^{\nu(s_1)}$, where

$$s_1 = \min(s, s_0 + \sigma).$$

By Theorem 1,

$$\varphi u \in H^{\mu\nu(s_1)},$$

i.e.,

$$u \in H_{\text{loc}}^{\mu\nu(s_1)}(\Omega).$$

Repeating this argument, we prove the theorem.

We shall make several concluding remarks.

- 1) The results of this work carry over trivially to systems of pseudodifferential operators constructed in the same way as hypoelliptic systems in ⁽¹²⁾.
- 2) Theorems 1 and 2 can be proved by the method of a priori estimates (see ^(12,13)).
- 3) One may consider symbols that are regular only with respect to a group of variables, and construct a theory of partially hypoelliptic pseudodifferential operators generalizing the corresponding results (see ⁽¹²⁻¹⁴⁾) for differential operators.
- 4) As Yu. V. Egorov pointed out to the author, if the regularizer is constructed by a more refined method ⁽¹⁵⁾, then hypoellipticity can also be proved for pseudodifferential operators of "variable strength."

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