

DOUBLE OPERATOR STIELTJES INTEGRALS AND MULTIPLIER PROBLEMS

M. Sh. Birman, M. Z. Solomyak

1966

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196601.46499>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.88+517.397+517.948

MATHEMATICS

M. Sh. BIRMAN, M. Z. SOLOMYAK

DOUBLE OPERATOR STIELTJES INTEGRALS AND MULTIPLIER PROBLEMS

(Presented by Academician V. I. Smirnov on 6 III 1966)

In a note by the authors ⁽¹⁾ (see also ⁽²⁾), properties were studied of double operator Stieltjes integrals of the form

$$Q \equiv \Phi T = \iint_{\Lambda M} \varphi(\lambda, \mu) F(d\mu) T E(d\lambda), \quad (1)$$

where φ is a complex function, T is a bounded operator in a Hilbert space $\mathfrak{H}$, and $F(\cdot)$ and $E(\cdot)$ are orthogonal spectral measures. Here we shall present further results in this direction, using the notation and terminology of ⁽¹⁾.

In Sec. 1 the properties are studied of the transformation (transformer) $Q = \Phi T$, acting in various symmetric normed ideals ⁽³⁾ of the ring $\mathfrak{A}$ of bounded operators in $\mathfrak{H}$. The corresponding results of ⁽¹⁾ are generalized by the fact that, first, we no longer assume Λ and M to be one-dimensional manifolds and, second, the smoothness conditions on the function φ are imposed not in the uniform metric but in the metric of the space L_p . Such generalizations are essential for applications.

In Sec. 2 the connection of integrals of the form (1) with a certain class of multiplier problems is clarified. In particular, our scheme includes the question of the boundedness of multidimensional singular integrals in L_2 (Sec. 3) and the trigonometric multiplier problem in l_p (Sec. 4). From the general theorems on transformers there follow both certain known and new results for these problems. Let us also note that, by writing a singular integral in the form (1), we give for it (apparently for the first time) an invariant definition.

1. Let $(\Lambda, \Sigma, E(\cdot))$, $(M, \Sigma_1, F(\cdot))$ be two spaces with orthogonal spectral measures. The definition of the integral (1), given in ⁽¹⁾ for $\Lambda = M = R^1$, carries over automatically to the case under consideration. Without dwelling on this question in detail, we point out only that, as in ⁽¹⁾, one can introduce a certain special orthogonal measure $\mathfrak{G}(\cdot)$ (essentially, the tensor product of the measures $E(\cdot)$ and $F(\cdot)$). The transformer Φ is then defined as the integral of the function $\varphi(\lambda, \mu)$ with respect to the measure

$\mathfrak{S}(\cdot)$. Below we shall assume the function $\varphi(\lambda, \mu)$ to be measurable and bounded with respect to the measure $\mathfrak{S}$:

$$(\mathfrak{S}) - \sup |\varphi(\lambda, \mu)| < \infty. \quad (2)$$

The analogue of Theorem 1 from ⁽¹⁾ is the following proposition.

Theorem 1. *Under condition (2), the integral (1) defines a transformer Φ of class $(\mathfrak{S}_2, \mathfrak{S}_2)$. The set of such transformers forms a commutative normed ring with involution, isomorphic and isometric to the ring of $\mathfrak{S}$ -measurable functions with norm (2).*

Denote by Q^m the unit cube in R^m , and by $W_p^\alpha(Q^m)$ the corresponding Sobolev-Slobodetskii functional space ⁽⁴⁾.

For $m = 1$ we shall also consider the space V_β (see, for example, ⁽⁵⁾) of functions of bounded β -variation.

Theorem 2. Let $\Lambda = Q^m$, $\varphi(\lambda, \mu) \in W_p^\alpha(Q^m)$ ($\alpha p > m$) as a function of λ for almost all $\mu \in M$, and suppose that

$$(F) - \sup \|\varphi(\cdot, \mu)\|_{W_p^\alpha} < \infty. \quad (3)$$

Then, for $p \leq 2$, the integral (1) generates a transformer belonging to each of the classes $(\mathfrak{S}_q, \mathfrak{S}_q)$, $1 \leq q \leq \infty$, and $(\mathfrak{R}, \mathfrak{R})$. For $m = 1$ this result is also true if, for almost all $\mu \in M$, $\varphi \in V_\beta$, $\beta < 2$, and $\varphi \in \text{Lip } \varepsilon$, $\varepsilon > 0$, with respect to the variable λ , and

$$(F) - \sup \{\|\varphi(\cdot, \mu)\|_{\text{Lip } \varepsilon} + \|\varphi(\cdot, \mu)\|_{V_\beta}\} < \infty. \quad (4)$$

Analogous assertions are valid when the roles of the variables λ and μ are interchanged.

We note that for $m = 1$ condition (4) is broader than condition (3).

Corollary 1. Under the hypotheses of Theorem 2, $\Phi \in (\mathfrak{S}, \mathfrak{S})$, where $\mathfrak{S}$ is any separable ⁽³⁾ (or conjugate to a separable) symmetric normed ideal in $\mathfrak{R}$.

Corollary 2. Under the hypotheses of Theorem 2, the transformer Φ induces on the factor-ring $\mathfrak{R}/\mathfrak{S}_\infty$ a bounded linear transformation whose norm does not exceed $\|\Phi\|_{1,1}$.

Theorem 3. Suppose that the hypotheses of the first part of Theorem 2 are satisfied, $p > 2$, and $\alpha \leq m/2$. Then the integral (1) generates a transformer $\Phi \in (\mathfrak{S}_q, \mathfrak{S}_q)$, where $|q^{-1} - 1/2| < \alpha m^{-1}$. If $\alpha = m/2$, then $\Phi \in (\mathfrak{S}_1, \mathfrak{S}_\infty)$ and $\Phi \in (\mathfrak{S}_\infty, \mathfrak{S}_\infty)^*$. If, in the hypotheses of the second part of Theorem 2, $\beta \geq 2$, then $\Phi \in (\mathfrak{S}_q, \mathfrak{S}_q)$ for $|q^{-1} - 1/2| < \beta^{-1}$.

Remark. The assertions of Theorems 2 and 3 remain valid if Λ is any smooth compact m -dimensional manifold, without boundary or with boundary.

We shall give brief indications concerning the method of proof of Theorems 2 and 3. Let $f, g \in \mathfrak{H}$, $\|f\| = \|g\| = 1$, and let the measures σ and τ be given by the formulas $\sigma(\cdot) = (E(\cdot)f, f)$, $\tau(\cdot) = (F(\cdot)g, g)$. According to the scheme set forth in ⁽²⁾, to prove Theorem 2 it suffices to estimate, uniformly with respect to f and g , the nuclear norm of the integral operator with kernel $\varphi(\lambda, \mu)$ acting from $L_2(\Lambda; \sigma)$ into $L_2(M; \tau)$. The required uniform estimates were obtained in the authors' paper ⁽⁵⁾. The proof of Theorem 3 uses the general idea of "interpolation by smoothness," applied earlier by Khirman ⁽⁶⁾ in a simpler situation. In carrying out the interpolation, it is necessary to make essential use of a number of results of ⁽⁵⁾, including Theorem 1 of ⁽⁵⁾, as well as the concrete method described there for approximating functions from W_p^α by piecewise polynomial functions. The proof of Corollary 1 is based on the interpolation theorem of B. S. Mityagin ⁽⁷⁾.

2. Let the space $\mathfrak{H}$ be decomposed into a direct integral

$$\mathfrak{H} = \int_{\Lambda} \oplus \mathfrak{H}(\lambda) \rho(d\lambda) \quad \left(\mathfrak{H} = \int_{\widetilde{M}} \oplus \mathfrak{H}_1(\mu) \rho_1(d\mu) \right) \quad (5)$$

so that the action of the operator $E(\delta)$, $\delta \subset \Lambda$ ($F(\partial)$, $\partial \subset M$), on an element $f \in \mathfrak{H}$ reduces to multiplication of the "representative" $f(\lambda)$ ($f_1(\mu)$) by the characteristic function of the set $\delta(\partial)$:

$$(E(\delta)f)(\lambda) = \chi_\delta(\lambda)f(\lambda) \quad ((F(\partial)f)_1(\mu) = \chi_\partial(\mu)f_1(\mu)).$$

To every operator $A \in \mathfrak{S}_2$, in the decompositions (5) there corresponds an operator kernel $A(\lambda, \mu)$, determined almost everywhere on $\Lambda \times M$, of class $\mathfrak{S}_2$, mapping

* For the definition of the ideals $\mathfrak{S}_\omega, \mathfrak{S}_\Omega$ see ⁽³⁾, §15.

$\mathfrak{H}(\lambda)$ into $\mathfrak{H}_1(\mu)$ and such that

$$(Af)_1(\mu) = \int_{\Lambda} A(\lambda, \mu) f(\lambda) \rho(d\lambda),$$

$$\|A\|_{\mathfrak{S}_2}^2 = \iint_{\Lambda M} \|A(\lambda, \mu)\|_{\mathfrak{S}_2}^2 \rho(d\lambda) \rho_1(d\mu).$$

It is easy to verify that, under the hypotheses of Theorem 1, the kernels corresponding to the operators T and Q are related by

$$Q(\lambda, \mu) = \varphi(\lambda, \mu)T(\lambda, \mu).$$

Denote by $\widetilde{\mathfrak{S}}_q$, $q \leq 2$, the class of kernels corresponding, in the decompositions (5), to operators of the class $\mathfrak{S}_q$. Starting from the relation $\mathfrak{S}_1^* = \mathfrak{R}$, one can associate a generalized kernel with every operator $A \in \mathfrak{R}$. Thus we arrive at the classes of generalized kernels $\widetilde{\mathfrak{R}}$, $\widetilde{\mathfrak{S}}_q$, $2 < q \leq \infty$, associated with the decompositions (5). The assertions of Theorems 2 and 3 can now be regarded as sufficient conditions for a scalar kernel $\varphi(\lambda, \mu)$ to be a multiplier in some of the classes $\widetilde{\mathfrak{S}}_q, \widetilde{\mathfrak{R}}$.

3. In this section $\mathfrak{H} = L_2(R^m)$, $\Lambda = R^m$, $M = S_{m-1}$ is the unit sphere in the space Ξ_m dual to R^m . If $\delta \subset S_{m-1}$, then $\delta(\subset \Xi_m)$ is the complete inverse image of δ under central projection. Let $F(\partial)$ be the operator of multiplication by the characteristic function of a set $\partial \subset R^m$, let $\mathcal{F}$ be the Fourier operator, and let $E(\delta) = \mathcal{F}^*F(\delta)\mathcal{F}$, $\delta \subset S_{m-1}$. Suppose further that a function $\sigma(\theta, x)$ ($\theta \in S_{m-1}, x \in R^m$) defines the transformer

$$\Phi = \iint \sigma(\theta, x)F(dx)(\cdot)E(d\theta)$$

of the class $(\mathfrak{R}, \mathfrak{R})$. We introduce the following definition.

Definition. A **singular integral (s.i.) operator** in $L_2(R^m)$ with symbol $\sigma(\theta, x)$ is the operator $J = \Phi I$ (I is the identity operator).

The considerations of § 2 make it possible easily to establish the equivalence of this definition and the customary definitions of an s.i. operator in terms of its symbol by means of the Fourier transform ⁽⁸⁻¹⁰⁾ or by means of a certain repeated integral ⁽⁸⁾. The advantage of the definition proposed here is its invariance and symmetry with respect to θ and x .

Theorem 2 now leads to various criteria for boundedness of s.i. operators in $L_2(R^m)$. We shall agree to write $\sigma \in \widetilde{W}_p^\alpha(S_{m-1})$ if $\sigma(\theta, x)$ belongs to W_p^α with respect to θ , and the norm of σ as a function of x is essentially (with respect to Lebesgue measure) bounded. Analogously for other functional classes, and also when the roles of θ and x are interchanged.

Theorem 4. Let J be an s.i. operator with symbol $\sigma(\theta, x)$, where $\sigma \in \widetilde{W}_2^\alpha(S_{m-1})$, $2\alpha > m - 1$, or, for $m = 2$, $\sigma \in \widetilde{V}_\beta(S_1) \cap \text{Lip } \varepsilon(S_1)$, $\beta < 2$, $\varepsilon > 0$. Then $J \in \mathfrak{R}$.

The first assertion of Theorem 4 was recently obtained, with the aid of expansions in spherical functions, by M. S. Agranovich ⁽⁹⁾, who refined the considerations of S. G. Mikhlin ⁽⁸⁾. The second assertion is apparently new.

Interchanging the roles of the variables x and θ leads to another type of boundedness conditions. Let $x = x(y)$ be the inverse stereographic mapping of the sphere S^m onto the extended space R^m .

Theorem 5. Let J be an s.i. operator with symbol $\sigma(\theta, x)$ and $\hat{\sigma}(\theta, y) = \sigma(\theta, x(y))$. If $\hat{\sigma} \in \widetilde{W}_2^\alpha(S^m)$, $2\alpha > m$, then $J \in \mathfrak{R}$.

If the symbol $\sigma(\theta, x)$ is finite with respect to x , then the assertion of Theorem 5 is valid under the condition $\sigma \in \widetilde{W}_2^\alpha(R^m)$, $2\alpha > m$. Imposing the latter condition locally, we arrive at a boundedness criterion for an s.i. operator in L_2 on a compact m -dimensional manifold.

The approach described makes results such as Theorem 5 quite natural. At the same time, until very recently the possibility of imposing requirements on the symbol with respect to the variable x , and not to θ , had not been noticed at all. Only recently, in a paper of Kohn and Nirenberg¹⁰, was a similar condition proposed in terms of the Fourier transform of the symbol with respect to the variable x . This condition and Theorem 5 do not cover one another, but, when formulated in terms of smoothness of the symbol, our conditions are less restrictive.

4. Let $\{c_k\}$, $k = (k_1, \dots, k_m)$, $-\infty < k_j < \infty$, be an m -fold sequence of class l_q , $q \geq 1$. Multiply the function

$$f(\lambda) = \sum_k c_k e^{i(k, \lambda)}, \quad \lambda = (\lambda_1, \dots, \lambda_m)$$

(for $q > 2$, possibly generalized) by a fixed periodic function $\psi(\lambda)$. Let Ψ be the linear transformation which sends the sequence $\{c_k\}$ to the sequence of Fourier coefficients of the function $\psi(\lambda)f(\lambda)$. It is required to indicate conditions under which Ψ is a bounded operator from l_q to l_q . The formulated trigonometric problem of multipliers^{11,6} is included in the scheme of § 2 if one sets $\Delta = M = \mathcal{T}_m$, where $\mathcal{T}_m$ is the m -dimensional torus; $\mathfrak{H} = L_2(\mathcal{T}_m)$; $E(\delta) = F(\delta)$ is the operator of multiplication by the characteristic function of the set $\delta \subset \mathcal{T}_m$; T is the integral operator in $L_2(\mathcal{T}_m)$ with kernel $f(\mu - \lambda)$; $\varphi(\lambda, \mu) = \psi(\mu - \lambda)$. Application of Theorems 2 and 3 leads to the following result.

Theorem 6. If $\psi \in W_p^\alpha(\mathcal{T}_m)$, $p\alpha > m$, then $\Psi \in (l_q, l_q)$, where $q \geq 1$ for $p \leq 2$ and $|q^{-1} - 1/2| < \alpha m^{-1}$ for $p > 2$, $2\alpha \leq m$.

As for the second assertion of Theorem 3, in the present case it leads to a result obtained earlier by Hirschman⁶ on another technical basis.

Leningrad State University
named after A. A. Zhdanov

Received
1 III 1966

REFERENCES

- ¹ M. Sh. Birman, M. Z. Solomyak, DAN, **165**, No. 6 (1965).
- ² M. Sh. Birman, M. Z. Solomyak, in the collection *Problems of Mathematical*

Physics, issue 1, L., 1966.

³ I. Ts. Gokhberg, M. G. Krein, *Introduction to the Theory of Linear Non-Selfadjoint Operators*, "Nauka," 1965.

⁴ S. M. Nikol'skii, UMN, **16**, issue 5 (101) (1961).

⁵ M. Sh. Birman, M. Z. Solomyak, DAN, **171**, No. 5 (1966).

⁶ I. I. Hirschman, Duke Math. J., **26**, No. 2 (1959).

⁷ B. S. Mityagin, Matem. sborn., **66** (108), 4 (1965).

⁸ S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, M., 1962.

⁹ M. S. Agranovich, UMN, **20**, issue 5 (125) (1965).

¹⁰ J. J. Kohn, L. Nirenberg, Comm. Pure and Appl. Math., **18**, No. 1/2 (1965).

¹¹ S. B. Stechkin, DAN, **71**, No. 3 (1950).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.