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Abstract

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MATHEMATICS

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ON UNIVALENT FUNCTIONS WITHOUT COMMON VALUES IN A MULTIPLY CONNECTED DOMAIN

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In the present note some new results are given for univalent functions without common values in a finitely connected domain, obtained from the consideration of suitable Dirichlet integrals.

Let B be a finite and finitely connected domain of the z -plane, bounded by closed analytic Jordan curves; $\mathcal{L}^2(B)$ the class of all functions regular and with integrable squared modulus in the domain B ; $\mathcal{L}_0^2(B)$ the subclass of functions from $\mathcal{L}^2(B)$ with single-valued integrals in the domain B ; $K(z, \zeta)$ and $l(z, \zeta)$ the Bergman kernels ⁽¹⁾ of the first and second kind of the domain B of the class $\mathcal{L}^2(B)$; $K_0(z, \zeta)$ and $l_0(z, \zeta)$ those of the class $\mathcal{L}_0^2(B)$;

$$U_k(z, \zeta) = \frac{1}{\pi} \left[\frac{f'_k(z)f'_k(\zeta)}{(f_k(z) - f_k(\zeta))^2} - \frac{1}{(z - \zeta)^2} \right], \quad z, \zeta \in B,$$

where the function $f_k(z)$ is regular* and univalent in the domain B .

Theorem 1. If the functions $f_k(z)$, $k = 1, \dots, n$, are regular, univalent, and without common values in the domain B , then for any points $\zeta_{k\mu}$ of this domain and any constants $\alpha_{k\mu}$, $\mu = 1, \dots, N$, $k = 1, \dots, n$, the inequality holds:

$$\left| \sum_{\mu, \nu=1}^N \left\{ \sum_{k=1}^n \alpha_{k\mu} \alpha_{k\nu} [U_k(\zeta_{k\mu}, \zeta_{k\nu}) + l_0(\zeta_{k\mu}, \zeta_{k\nu})] + \frac{2}{\pi} \sum_{1 \leq j < k \leq n} \alpha_{j\mu} \alpha_{k\nu} \frac{f'_j(\zeta_{j\mu}) f'_k(\zeta_{k\nu})}{(f_j(\zeta_{j\mu}) - f_k(\zeta_{k\nu}))^2} \right\} \right| \leq \sum_{\mu, \nu=1}^N \sum_{k=1}^n \alpha_{k\mu} \bar{\alpha}_{k\nu} K_0(\zeta_{k\mu}, \zeta_{k\nu}) \quad (1^0)$$

This inequality remains valid also after the simultaneous replacement in it of the functions l_0 and K_0 , respectively, by the functions l and K (this second inequality will henceforth be cited as inequality (1)).

One can indicate necessary and sufficient conditions under which equality holds in the inequalities (1^0) and (1).

Each of the inequalities (1⁰) and (1) determines the corresponding disk in which lie the possible values of the functional

$$\sum_{\mu, \nu=1}^N \left[\sum_{k=1}^n \alpha_{k\mu} \alpha_{k\nu} U_k(\zeta_{k\mu}, \zeta_{k\nu}) + \frac{2}{\pi} \sum_{1 \leq j < k \leq n} \alpha_{j\mu} \alpha_{k\nu} \frac{f'_j(\zeta_{j\mu}) f'_k(\zeta_{k\nu})}{(f_j(\zeta_{j\mu}) - f_k(\zeta_{k\nu}))^2} \right]$$

for fixed $\zeta_{k\mu}$ and $\alpha_{k\mu}$, $\mu = 1, \dots, N$; $k = 1, \dots, n$.

The disk determined by inequality (1⁰) lies in the disk determined by inequality (1).

For $n = 1$, inequalities (1⁰) and (1) give inequalities previously obtained by another method by Bergman and Schiffer (1).

Corollary 1. If the function $f(z)$ is regular, univalent, and bounded in the domain B : $|f(z)| < 1$, $z \in B$, then for any points ζ_μ of this domain

* Here and below—including single-valuedness.

and any constants α_μ , $\mu = 1, \dots, N$, the inequality holds:

$$\left| \sum_{\mu, \nu=1}^N \alpha_\mu \alpha_\nu [U(\xi_\mu, \xi_\nu) + l_0(\xi_\mu, \xi_\nu)] \right| \leq \sum_{\mu, \nu=1}^N \alpha_\mu \bar{\alpha}_\nu \left[K_0(\xi_\mu, \bar{\xi}_\nu) - \frac{f'(\xi_\mu) \overline{f'(\xi_\nu)}}{\pi(1 - f(\xi_\mu) \overline{f(\xi_\nu)})^2} \right]. \quad (2)$$

In particular:

$$\left| \frac{1}{6} \{f, z\} + \pi l_0(z, z) \right| + \frac{|f'(z)|^2}{(1 - |f(z)|^2)^2} \leq \pi K_0(z, \bar{z}), \quad z \in B,$$

where $\{f, z\}$ is the Schwarzian invariant.

Inequality (2) strengthens Singh's inequality (2) for bounded univalent functions, in which l_0 and K_0 are replaced by l and K .

Let $\widetilde{C}(B)$ denote the class of all functions $f(z)$, regular and univalent in the domain B , satisfying the condition: $f(z_1)f(z_2) \neq 1$ for any points z_1 and z_2 of the domain B .

Corollary 2. If $f(z) \in \widetilde{C}(B)$, then for any points ξ_μ of the domain B and any constants α_μ , $\mu = 1, \dots, N$, the inequality holds:

$$\left| \sum_{\mu, \nu=1}^N \alpha_\mu \alpha_\nu \left[U(\xi_\mu, \xi_\nu) \pm \frac{f'(\xi_\mu) f'(\xi_\nu)}{\pi(1 - f(\xi_\mu) f(\xi_\nu))^2} + l_0(\xi_\mu, \xi_\nu) \right] \right| \leq \sum_{\mu, \nu=1}^N \alpha_\mu \bar{\alpha}_\nu K_0(\xi_\mu, \bar{\xi}_\nu),$$

In particular:

$$\left| \frac{1}{6} \{f, z\} \pm \left(\frac{f'(z)}{1 - f^2(z)} \right)^2 + \pi l_0(z, z) \right| \leq \pi K_0(z, \bar{z}), \quad z \in B.$$

Each of these inequalities remains valid after replacing l_0 and K_0 in it respectively by l and K .

Theorem 2. Let the functions $f_k(z)$, $k = 1, \dots, n$, be regular in a domain B containing the origin and satisfy the conditions: $f_j(0) \neq f_k(0)$, $j \neq k$, $f'_k(0) \neq 0$. In order that these functions be univalent functions without common values in the domain B , it is necessary and sufficient that, for any n complex vectors $\{\alpha_{k0}, \dots, \alpha_{kN}\}$, $k = 1, \dots, n$, with arbitrary $N = 0, 1, 2, \dots$, the conditions

$$\sum_{\mu, \nu=0}^N \left[\sum_{k=1}^n \alpha_{k\mu} \alpha_{k\nu} (c_{\mu\nu}^{[k]} + \lambda_{\mu\nu}) + \sum_{1 \leq j < k \leq n} \alpha_{j\mu} \alpha_{k\nu} d_{\mu\nu}^{[jk]} \right] \leq \sum_{\mu, \nu=0}^N \varkappa_{\mu\nu} \sum_{k=1}^n \alpha_{k\mu} \bar{\alpha}_{k\nu},$$

be satisfied, where $c_{\mu\nu}^{[k]}$, $d_{\mu\nu}^{[jk]}$, $\lambda_{\mu\nu}$, and $\varkappa_{\mu\nu}$ are determined by the following expansions into double series in a neighborhood of the origin:

$$\begin{aligned} U_k(z, \zeta) &= \sum_{\mu, \nu=0}^{\infty} c_{\mu\nu}^{[k]} z^\mu \zeta^\nu, \quad k = 1, \dots, n; \\ \frac{2f'_j(z)f'_k(\zeta)}{\pi[f_j(z) - f_k(\zeta)]^2} &= \sum_{\mu, \nu=0}^{\infty} d_{\mu\nu}^{[jk]} z^\mu \zeta^\nu, \quad 1 \leq j < k \leq n; \\ l(z, \zeta) &= \sum_{\mu, \nu=0}^{\infty} \lambda_{\mu\nu} z^\mu \zeta^\nu, \quad K(z, \bar{\zeta}) = \sum_{\mu, \nu=0}^{\infty} \varkappa_{\mu\nu} z^\mu \bar{\zeta}^\nu. \end{aligned}$$

For $n = 1$ this theorem gives the known ⁽¹⁾ necessary and sufficient conditions for univalence of a function in a multiply connected domain.

Corollary. Let the function $f(z)$ be regular in the domain B , containing the origin, and satisfy the conditions $f(0) \neq \pm 1$, $f'(0) \neq 0$. In order that the function $f(z)$ belong to the class $\widetilde{C}(B)$, it is necessary and sufficient that, for any two complex vectors $\{a_{k0}, \dots, a_{kN}\}$, $k = 1, 2$, with arbitrary $N = 0, 1, 2, \dots$, the conditions

$$\left| \sum_{\mu, \nu=0}^N [(\alpha_{1\mu} \alpha_{1\nu} + \alpha_{2\mu} \alpha_{2\nu})(c_{\mu\nu} + \lambda_{\mu\nu}) - \alpha_{1\mu} \alpha_{2\nu} d_{\mu\nu}] \right| \leq \sum_{\mu, \nu=0}^N (\alpha_{1\mu} \bar{\alpha}_{1\nu} + \alpha_{2\mu} \bar{\alpha}_{2\nu}) \chi_{\mu\nu},$$

hold, where the coefficients $c_{\mu\nu}$, $\lambda_{\mu\nu}$, and $\chi_{\mu\nu}$ are defined in Theorem 2, while $d_{\mu\nu}$ are determined by the expansion in a neighborhood of the origin

$$\frac{2f'(z)f'(\zeta)}{\pi[1-f(z)f(\zeta)]^2} = \sum_{\mu,\nu=0}^{\infty} d_{\mu\nu} z^{\mu} \zeta^{\nu}.$$

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Note: Figure translations are in progress. See original paper for figures.

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