

ON CONSERVATION LAWS IN EINSTEIN' S THEORY OF GRAVITATION

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Abstract

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PHYSICS

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ON CONSERVATION LAWS IN EINSTEIN' S THEORY OF GRAVITATION

(Presented by Academician V. A. Fock on 11 XII 1965)

The paper considers the connection between conservation laws in Einstein' s theory of gravitation and the properties of the Lagrangian function of the gravitational field.

We shall start from the Einstein gravitational equations in Lagrangian form:

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{\partial}{\partial x_\alpha} \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu} / \partial x_\alpha)} = -\kappa \sqrt{-g} T^{\mu\nu}, \quad (1)$$

where

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} (\Gamma_{\mu\alpha}^\beta \Gamma_{\nu\beta}^\alpha - \Gamma_{\mu\nu}^\alpha \Gamma_{\alpha\beta}^\beta) \quad (2)$$

is the Lagrangian function of the gravitational field; $g_{\mu\nu}$ is the fundamental tensor; g is the determinant composed of the components of the fundamental tensor $g_{\mu\nu}$; $\Gamma_{\mu\nu}^\alpha$ are Christoffel symbols of the second kind; κ is Einstein' s gravitational constant, and $T^{\mu\nu}$ is the mass tensor. Greek indices take the values 0, 1, 2, 3. Summation over identical Greek indices from 0 to 3 is assumed.

For the purpose of deriving conservation laws, following works ⁽¹⁻⁴⁾, let us pass from the variables x_0, x_1, x_2, x_3 to new variables x'_0, x'_1, x'_2, x'_3 , putting

$$x'_\beta = x_\beta(x_0, x_1, x_2, x_3, \varepsilon), \quad (3)$$

where ε is a certain parameter, and

$$(x'_\beta)_{\varepsilon=0} = x_\beta. \quad (4)$$

Taking into account that the Lagrangian function (2) depends only on $g_{\mu\nu}$ and their first derivatives with respect to the variables x_α , and noting that

$$\left(\frac{\partial}{\partial \varepsilon} \frac{\partial g'_{\mu\nu}}{\partial x'_\alpha}\right)_{\varepsilon=0} = \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}, \quad (5)$$

we obtain

$$\begin{aligned} \left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon}\right)_{\varepsilon=0} &= \mathcal{L} \frac{\partial}{\partial x_\beta} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} + \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} \\ &\quad - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}. \end{aligned} \quad (6)$$

Here

$$\mathcal{L}^* = \mathcal{L}' \left| \frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} \right|, \quad (7)$$

$$\mathcal{L}' = \sqrt{-g'} g'^{\mu\nu} (\Gamma'_{\mu\alpha}{}^\beta \Gamma'_{\nu\beta}{}^\alpha - \Gamma'_{\mu\nu}{}^\alpha \Gamma'_{\alpha\beta}{}^\beta). \quad (2')$$

If we now make use of the gravitational equations (1) and the equalities

$$\left(\frac{\partial g'_{\mu\nu}}{\partial \varepsilon}\right)_{\varepsilon=0} = -g_{\mu\cdot} \frac{\partial}{\partial x_\nu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} - g_{\nu\cdot} \frac{\partial}{\partial x_\mu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0}, \quad (8)$$

then we shall have

$$\begin{aligned} \left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon}\right)_{\varepsilon=0} &= \left(\mathcal{L} \delta_\beta^\alpha - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu}/\partial x_\alpha)} \right) + 2\kappa \sqrt{-g} g T_\beta^\alpha \frac{\partial}{\partial x_\alpha} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} \\ &\quad - 2 \frac{\partial}{\partial x_\alpha} \left[g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial (\partial g_{\mu\nu}/\partial x_\alpha)} \frac{\partial}{\partial x_\nu} \left(\frac{\partial x'_\beta}{\partial \varepsilon}\right)_{\varepsilon=0} \right], \end{aligned} \quad (9)$$

where $\delta_\beta^\alpha = 1$ for $\alpha = \beta$ and $\delta_\beta^\alpha = 0$ for $\alpha \neq \beta$. Consequently, taking into account Einstein's conservation laws,

$$\frac{\partial}{\partial x_\alpha} \sqrt{-g} (T_\beta^\alpha + \theta_\beta^\alpha) = 0, \quad (10)$$

where

$$\theta_{\beta}^{\alpha} = \frac{1}{2\kappa\sqrt{-g}} \left(\mathcal{L}\delta_{\beta}^{\alpha} - \frac{\partial g_{\mu\nu}}{\partial x_{\beta}} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \right) \quad (11)$$

is Einstein' s energy-momentum pseudotensor, we arrive at the conclusion that

$$\left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon} \right)_{\varepsilon=0} = \frac{\partial}{\partial x_{\alpha}} \left\{ 2\kappa\sqrt{-g} (T_{\beta}^{\alpha} + \theta_{\beta}^{\alpha}) \left(\frac{\partial x'_{\beta}}{\partial \varepsilon} \right)_{\varepsilon=0} - 2g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \frac{\partial}{\partial x_{\nu}} \left(\frac{\partial x'_{\beta}}{\partial \varepsilon} \right)_{\varepsilon=0} \right\}. \quad (12)$$

In deriving equality (12), the equations of gravitation (1) were essentially used. We shall now calculate the same quantity $(\partial \mathcal{L}^*/\partial \varepsilon)_{\varepsilon=0}$, but without taking account of the equations of gravitation. For this purpose we shall use the transformation law of the Lagrangian function (2). As is known (see (5)),

$$\mathcal{L}' \left| \frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} \right| = \mathcal{L} + \frac{\partial}{\partial x_{\alpha}} (P_{\mu\nu}^{\alpha} \mathfrak{G}^{\mu\alpha} - P_{\mu\nu}^{\alpha} \mathfrak{G}^{\mu\nu}), \quad (13)$$

$$P_{\mu\nu}^{\alpha} = \frac{\partial x'_{\sigma}}{\partial x_{\mu}} \frac{\partial x'_{\tau}}{\partial x_{\nu}} \frac{\partial^2 x_{\alpha}}{\partial x'_{\sigma} \partial x'_{\tau}}, \quad \mathfrak{G}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}. \quad (14)$$

Since

$$\left(\frac{\partial P_{\mu\nu}^{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0} = - \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} \left(\frac{\partial x'_{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0}, \quad (15)$$

we have, according to (13),

$$\left(\frac{\partial \mathcal{L}^*}{\partial \varepsilon} \right)_{\varepsilon=0} = - \frac{\partial}{\partial x_{\alpha}} \left\{ \frac{\partial \mathfrak{G}^{\mu\nu}}{\partial x_{\mu}} \frac{\partial}{\partial x_{\nu}} \left(\frac{\partial x'_{\alpha}}{\partial \varepsilon} \right)_{\varepsilon=0} - \frac{\partial \mathfrak{G}^{\mu\alpha}}{\partial x_{\nu}} \frac{\partial}{\partial x_{\mu}} \left(\frac{\partial x'_{\nu}}{\partial \varepsilon} \right)_{\varepsilon=0} \right\}. \quad (16)$$

Comparing (12) and (16), we arrive at the following conservation laws

$$\frac{\partial}{\partial x_{\alpha}} \left\{ 2\kappa\sqrt{-g} (T_{\beta}^{\alpha} + \theta_{\beta}^{\alpha}) \varphi^{\beta} - 2g_{\mu\beta} \frac{\partial \mathcal{L}}{\partial(\partial g_{\mu\nu}/\partial x_{\alpha})} \frac{\partial \varphi^{\beta}}{\partial x_{\nu}} + \frac{\partial \mathfrak{G}^{\mu\nu}}{\partial x_{\mu}} \frac{\partial \varphi^{\alpha}}{\partial x_{\nu}} - \frac{\partial \mathfrak{G}^{\nu\alpha}}{\partial x_{\mu}} \frac{\partial \varphi^{\beta}}{\partial x_{\nu}} \right\} = 0, \quad (17)$$

$$\varphi^{\alpha} = (\partial x'_{\alpha}/\partial \varepsilon)_{\varepsilon=0}$$

are arbitrary functions of the variables x_0, x_1, x_2, x_3 . If arbitrary constants are chosen as φ^α , then from (17) Einstein's conservation laws (10) follow at once. Setting in (17) $\varphi^\alpha = c_\beta \mathfrak{G}^{\alpha\beta}$, where c_β are arbitrary constants, we obtain the conservation laws considered in ⁽⁶⁾. If, however, using the equations of gravitation, one eliminates the mass tensor from (17), then one arrives at the conservation laws considered in ^(4,7,8).

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Note: Figure translations are in progress. See original paper for figures.

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