

INTEGRAL GEOMETRY ON THE MANIFOLD OF $\backslash(k\backslash)$ -DIMENSIONAL PLANES

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Abstract

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MATHEMATICS

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INTEGRAL GEOMETRY ON THE MANIFOLD OF k -DIMENSIONAL PLANES

Consider the set $H_{n,k}$ of k -dimensional planes of the n -dimensional complex space C^n . On each plane $h \in H_{n,k}$ let us specify a measure μ_h , invariant with respect to parallel translations. Further, let $f(x)$ be an infinitely differentiable and rapidly decreasing function in C^n . The formula

$$\varphi(h) = \int_h f(x) d\mu_h \quad (1)$$

assigns to each such function $f(x)$ a certain function of the plane, $\varphi(h)$. The aim of the present work is to invert formula (1). Since the manifold $H_{n,k}$ has dimension greater than n (namely $(k+1)(n-k)$), in order to determine $f(x)$ it is natural to use the values of the function $\varphi(h)$ (and its derivatives) only on some n -dimensional submanifold of $H_{n,k}$. It will also be clarified which class of functions on $H_{n,k}$ is determined by formula (1).

1. We shall assume that the measure μ_h satisfies the following two conditions:
 - 1) if $A \subset h$ is a set of positive measure and g is a parallel translation in C^n , then $\mu_{hg}(Ag) = \mu_h(A)$; 2) if g is an arbitrary affine transformation in C^n , then $\mu_{hg}(Ag)$ is a continuous infinitely differentiable function of g^* .

Denote by $G_{n,k}$ the complex Grassmann manifold of k -dimensional subspaces of the space C^n (i.e., k -dimensional planes passing through the point O). Consider the mapping

$$\pi : H_{n,k} \rightarrow G_{n,k},$$

which assigns to each plane $h \in H_{n,k}$ the plane from $G_{n,k}$ parallel to it. This mapping endows $H_{n,k}$ with the structure of a fiber space whose base is $G_{n,k}$, and whose fiber is the set of mutually parallel k -dimensional planes. For each point $\beta \in C^n$ we further introduce the mapping

$$s_\beta : G_{n,k} \rightarrow H_{n,k},$$

which assigns to each plane from $G_{n,k}$ the parallel plane passing through the point β . Obviously, the composition $\pi \circ s_\beta$ is the identity mapping of the base onto itself.

Denote by Φ_H the set of all infinitely differentiable functions on $H_{n,k}$, and by $\Phi_H^{(p,q)}$ the space of forms of type (p,q) on $H_{n,k}$ with coefficients from Φ_H . Analogously define Φ_G and $\Phi_G^{(p,q)}$. The mappings π and s_β induce mappings of functions (respectively, of differential forms)

$$\pi^* : \Phi_G \rightarrow \Phi_H, \quad \Phi_G^{(p,q)} \rightarrow \Phi_H^{(p,q)};$$

$$s_\beta^* : \Phi_H \rightarrow \Phi_G, \quad \Phi_H^{(p,q)} \rightarrow \Phi_G^{(p,q)}.$$

* Such a compatible set of measures can be introduced, for example, by specifying a Hermitian metric in C^n and defining $\mu_h(A)$ as the measure in this metric.

2. In this section we shall construct the operator κ_ρ , which plays the principal role in the inversion formula and maps Φ_H into $\Phi_G^{(k,k)}$. Before defining this operator, let us consider the following algebraic construction.

Introduce the space $S^k(C^n) \otimes \bar{S}^k(C^n)$, where $S^k(C^n)$ is the set of symmetric polylinear forms of the first kind of degree k in the space C^n , and $\bar{S}^k(C^n)$ is the set of analogous forms of the second kind. An arbitrary element of this space may be represented in the form

$$B(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k), \quad \xi_\nu, \xi'_\nu \in C^n,$$

where B is a form of degree $2k$ with complex coefficients, symmetric both in the first and in the second group of arguments, linear in the arguments ξ_ν and antilinear in the arguments ξ'_ν . Now consider two square matrices of order k , whose elements are the vectors $\xi_{ij} \in C^n$ and $\xi'_{ij} \in C^n$ ($i, j = 1, \dots, k$). To each form $B(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k)$ and to the matrices $\Xi = \|\xi_{ij}\|$, $\Xi' = \|\xi'_{ij}\|$ we associate the new form

$$\text{Det}_B(\Xi, \Xi') = \sum_{\sigma, \sigma'} (-1)^{\mu+\nu} B(\xi_{1,s_1}, \dots, \xi_{k,s_k}; \xi'_{1,t_1}, \dots, \xi'_{k,t_k}),$$

where the summation is over all permutations $\sigma(s_1, \dots, s_k)$, $\sigma'(t_1, \dots, t_k)$ of the numbers $(1, 2, \dots, k)$, and μ and ν are the numbers of inversions in the permutations σ and σ' . It is easy to see that, like the ordinary numerical determinant, this “determinant of vectors” is skew-symmetric with respect to the rows and with respect to the columns of the matrices Ξ and Ξ' .

Lemma 1. *If the form $B(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k)$ is defined in the quotient space C^n by some subspace E^* and the k vectors $\xi_{i1}, \dots, \xi_{ik}$ or $\xi'_{i1}, \dots, \xi'_{ik}$, for some fixed i , belong to E , then $\text{Det}_B(\Xi, \Xi') = 0$.*

We now proceed to the construction of the operator $\varkappa_\rho$. Let $\varphi(h) \in \Phi_H$. Take an arbitrary fixed point $\beta \in C^n$. The transformation s_β^* takes the function $\varphi(h)$ into a function on $G_{n,k}$, depending on the point β as on a parameter. Put $s_\beta^* \varphi(h) = \varphi(\alpha; \beta)$, where $\alpha = \pi h$, i.e. α is the plane in $G_{n,k}$ parallel to h . Clearly, if β and β' are different points of the plane h , then $\varphi(\alpha; \beta) = \varphi(\alpha; \beta')$. Define now, by means of the function φ , the form

$$\begin{aligned} B_\varphi(\xi_1, \dots, \xi_k; \xi'_1, \dots, \xi'_k) &= d_\beta^{k,k} \varphi(\alpha; \beta) = \\ &= \sum_{\substack{p_1, \dots, p_k \\ q_1, \dots, q_k}} \frac{\partial^{2k} \varphi(\alpha; \beta)}{\partial \beta_{p_1} \dots \partial \beta_{p_k} \partial \bar{\beta}_{q_1} \dots \partial \bar{\beta}_{q_k}} \xi_1^{p_1} \dots \xi_k^{p_k} \bar{\xi}_1^{q_1} \dots \bar{\xi}_k^{q_k}. \end{aligned}$$

Here the upper index is the number of a coordinate of the vector. The summation is over the values of each of the upper indices from 1 to n . Next assign to an arbitrary point $\alpha \in G_{n,k}$ a frame consisting of k linearly independent vectors $\alpha_j \in C^n$ ($j = 1, \dots, k$). Consider k arbitrary displacements of this frame and put $\xi_{ij} = \xi'_{ij} = d_i \alpha_j$ ($i, j = 1, \dots, k$). Define the operator $\varkappa_\rho$ by the formula

$$\begin{aligned} \varkappa_\rho \varphi(h) &= \left(\frac{i}{2}\right)^k \text{Det}_{d_\beta^{k,k} \varphi}(\Xi, \Xi') = \\ &= \left(\frac{i}{2}\right)^k \sum_{\sigma, \sigma'} (-1)^{\mu+\nu} \sum_{\substack{p_1, \dots, p_k \\ q_1, \dots, q_k}} \frac{\partial^{2k} \varphi(\alpha; \beta)}{\partial \beta_{p_1} \dots \partial \beta_{p_k} \partial \bar{\beta}_{q_1} \dots \partial \bar{\beta}_{q_k}} \times \\ &\quad \times d_1 \alpha_{s_1}^{p_1} \wedge \dots \wedge d_k \alpha_{s_k}^{p_k} \wedge d_1 \bar{\alpha}_{t_1}^{q_1} \wedge \dots \wedge d_k \bar{\alpha}_{t_k}^{q_k}. \end{aligned}$$

* The form B is defined in the quotient space C^n/E if it vanishes whenever at least one of the vectors ξ_ν belongs to E .

Here σ, σ' , as above, are permutations of the numbers $(1, 2, \dots, k)$, and μ and ν are the numbers of inversions in these permutations. Formally, the form $\chi_{\beta\varphi}$ depends on k^2 vectors from C^n . However, when the increment of the vector β lies in the subspace $\alpha \subset C^n$, then $d_\beta^{k,k} \varphi(\alpha; \beta) = 0$, i.e., the form $d_\beta^{k,k} \varphi(\alpha; \beta)$ is defined in the quotient space C^n/α . Hence, and from Lemma 1, it follows easily that the form $\chi_{\beta\varphi}$ is defined in the tangent space to $G_{n,k}$ at the point a , i.e., the following holds.

Lemma 2. The form $\chi_{\beta\varphi}(h)$ belongs to $\Phi_G^{(k,k)}$.

3. We proceed to the formulation of the main theorems.

Theorem 1. In order that a function $\varphi(h)$ on $H_{n,k}$, where $k < n - 1$, be representable in the form (1), where $f(x)$ is a rapidly decreasing infinitely differentiable function on C^n , it is necessary and sufficient that the following conditions hold:

- 1) $\varphi(h) \in \Phi_H$; for every fixed α , the function $s_\beta^* \varphi(h) = \varphi(\alpha; \beta)$ is a rapidly decreasing function in the space C^n / α^* .
- 2) The form $\chi_{\beta\varphi}(h)$ is closed in $\Phi_G^{(k,k)}$ **.

Theorem 2. If $\varphi(h) = \int_h f(x) d\mu_h$, and γ is an arbitrary cycle in $G_{n,k}$ of real dimension $2k$, then

$$\int_\gamma \chi_{\beta\varphi} = c_\gamma f(\beta), \quad (2)$$

where the constant c_γ depends only on the homology class of the cycle γ .

Let $\gamma_0, \gamma_1, \dots, \gamma_\nu$ be a basis of the $2k$ -dimensional homology group of the complex Grassmann manifold $G_{n,k}$. It is known ⁽¹⁾ that each of the cycles γ_s is determined by a partition of the number k into a sum of k nonincreasing integers $a_1 + a_2 + \dots + a_k = k$, $0 \leq a_1 \leq \dots \leq a_k \leq n - k$. In particular, the cycle γ_0 , corresponding to the partition $1 + 1 + \dots + 1 = k$, consists of all k -dimensional subspaces $G_{n,k}$ belonging to a fixed $(k + 1)$ -dimensional subspace, i.e. $\gamma_0 \approx G_{k+1,k}$.

Theorem 3. $c_{\gamma_0} = (-1)^k \pi^{2k} / (k!)^2$; $c_{\gamma_s} = 0$, $s = 1, \dots, \nu$.

The inversion formula (2) can be rewritten in a somewhat different form. Let us specify a k -dimensional plane h by equations of the form

$$P_i(x) \equiv x_i - \alpha_i^1 x_{l+1} - \dots - \alpha_i^k x_n - \beta_i = 0, \quad i = 1, 2, \dots, l,$$

where $l = n - k$. Define the functional $\varphi(h)$ by the formula

$$\varphi(h) = (\delta(P_1, \dots, P_l), f(x)) \quad ***.$$

Then the inversion formula (2) is equivalent to the relation

$$\begin{aligned} & \left(\frac{i}{2}\right)^k \sum_{\sigma, \sigma'} (-1)^{\mu+\nu} \int_\gamma \sum_{\substack{p_1, \dots, p_k \\ q_1, \dots, q_k}} \frac{\partial^{2k} \delta(P_1 - P_1^0, \dots, P_l - P_l^0)}{\partial P_{p_1} \dots \partial P_{p_k} \partial \bar{P}_{q_1} \dots \partial \bar{P}_{q_k}} \times \\ & \times d_1 \alpha_{s_1}^{p_1} \wedge \dots \wedge d_k \alpha_{s_k}^{p_k} \wedge d_1 \bar{\alpha}_{t_1}^{q_1} \wedge \dots \wedge d_k \bar{\alpha}_{t_k}^{q_k} = c_\gamma \delta(x - x_0), \end{aligned}$$

where $x \in C^n$, x_0 is a fixed point in C^n , $P_i^0 = P_i(x_0)$.

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* As was already noted, it follows from the definition of $\varphi(\alpha; \beta)$ that, as a function of β , it is constant on the cosets C^n/α .

** For $k = n - 1$ the condition formulated above is insufficient. For a formulation of the necessary and sufficient conditions for $k = n - 1$, see (2).

*** $\delta(P_1, \dots, P_l)$ may be defined as the product of generalized functions $\delta(P_1), \dots, \delta(P_l)$; for the definition of the function $\delta(P)$, see (2).

Note: Figure translations are in progress. See original paper for figures.

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