

ON SHARP ESTIMATES OF EIGENFUNCTIONS OF THE BIHARMONIC OPERATOR

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Abstract

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MATHEMATICS

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ON SHARP ESTIMATES OF EIGENFUNCTIONS OF THE BIHARMONIC OPERATOR

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In this note we establish sharp estimates in a closed domain for eigenfunctions of the biharmonic operator and their derivatives. It is known how important such estimates are in questions of expansion in eigenfunctions of self-adjoint operators.

The problem under consideration is as follows: let g be an arbitrary bounded m -dimensional domain with boundary Γ ; let $\{u_n(x)\}$ be orthonormal eigenfunctions; and let $\{\lambda_n\}$ be the corresponding eigenvalues of the problem

$$\Delta^2 u - \lambda u = 0 \quad \text{in the domain } g; \quad u|_{\Gamma} = \partial u / \partial \nu|_{\Gamma} = 0. \quad (1)$$

Here $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator; ν is the outward normal to the boundary Γ ; $\lambda_n > 0$; $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. It is required to find an estimate of

$$\max_{x \in (g+\Gamma)} |u_n(x)|.$$

If one takes the m -dimensional ball as the domain g , then direct computation shows that the eigenfunctions of problem (1) possessing radial symmetry take, at the center of the ball—the point x_0 —the values $|u_n(x_0)| = C_n \lambda_n^{(m-1)/8}$, where $C_n \geq C > 0$. Therefore, in the case of an arbitrary domain g , an estimate of the maximum modulus of the eigenfunctions of the form*

$$\max_{x \in (g+\Gamma)} |u_n(x)| = O[f(\lambda_n)] \quad (2)$$

is possible only if $f(\lambda_n) \geq \lambda_n^{(m-1)/8}$.

Here I prove that for eigenfunctions of problem (1) in an arbitrary closed domain $g + \Gamma$ the sharp estimate (2) is valid with $f(\lambda_n) = \lambda_n^{(m-1)/8}$, i.e. the estimate

$$\max_{x \in (g+\Gamma)} |u_n(x)| = O\left(\lambda_n^{(m-1)/8}\right). \quad (3)$$

Obtaining this estimate in any interior closed subdomain g' of the domain g presents no serious difficulties, since here one can use considerations analogous to those applied by Titchmarsh in deriving sharp interior estimates for eigenfunctions of the Laplace operator (see ⁽¹⁾). Estimates in a closed domain require an entirely different approach. The method I use is close to that applied in works ^(2, 3) in the study of eigenfunctions of the Laplace operator.

We shall assume that the domain $(g + \Gamma)$ belongs to the class** $A^{(4)}$.

Lemma 1. *For the eigenfunctions of problem (1) the following integral estimates over the domain g hold:*

$$\int_g \sum_{i=1}^m \left(\frac{\partial u_n}{\partial x_i} \right)^2 dx = O\left(\lambda_n^{1/2}\right), \quad \int_g (\Delta u_n)^2 dx = O(\lambda_n), \quad \int_g \sum_{i,k=1}^m \left(\frac{\partial^2 u_n}{\partial x_i \partial x_k} \right)^2 dx = O(\lambda_n). \quad (4)$$

* The constants entering the O -term estimates here and below do not depend on n .

** This means that the function defining the equation of the surface Γ in local coordinates belongs to the class $A^{(4)}$.

Lemma 1 is a trivial consequence of conditions (1) and of Green's first and second formulas.

Lemma 2. *For the eigenfunctions of problem 1 the following integral estimates over the boundary Γ are valid:*

$$\int_{\Gamma} (\Delta u_n)^2 ds = O(\lambda_n), \quad \int_{\Gamma} \left[\frac{\partial}{\partial \nu} (\Delta u_n) \right]^2 ds = O(\lambda_n^{3/2}). \quad (5)$$

The proof of this lemma is cumbersome and cannot be given here in full; we indicate only the main points. The first estimate of the lemma is obtained simply: for any eigenfunction the equality

$$\begin{aligned} \int_{\Gamma} (\Delta u_n)^2 ds &= 2 \int_g \Delta u_n \sum_{i=1}^m \frac{\partial(\Delta u_n)}{\partial x_i} q_i dx + \int_g (\Delta u_n)^2 \sum_{i=1}^m \frac{\partial q_i}{\partial x_i} dx = \\ &= 2 \int_g u_n \Delta \left(\sum_{i=1}^m \frac{\partial(\Delta u_n)}{\partial x_i} q_i \right) dx + \int_g (\Delta u_n)^2 \sum_{i=1}^m \frac{\partial q_i}{\partial x_i} dx, \end{aligned} \quad (6)$$

is valid, where $q_i(x)$ are smooth functions satisfying the condition $q_i(x)|_{x \in \Gamma} = \cos(\nu, x_i)$. The right-hand side of formula (6) is easily estimated with the aid of integration by parts, Lemma 1, and conditions (1). As a result we obtain the required estimate

$$\int_{\Gamma} [(\Delta u_n)^2] ds = O(\lambda_n).$$

The proof of the second assertion of Lemma 2 is much more complicated. Its idea is as follows. First, using Green's first formula, we obtain the equality

$$\int_{\Gamma} \left[\frac{\partial}{\partial \nu} (\Delta u_n) \right]^2 ds = \int_{\Gamma} [\nabla_{\tau} (\Delta u_n)]^2 ds + O \left(\int_g [\nabla (\Delta u_n)]^2 dx \right) + O(\lambda_n^{3/2}), \quad (7)$$

where ∇_{τ} is the gradient in the hyperplane tangent to Γ ; ∇ is the gradient. Then, locally straightening the boundary Γ and introducing a cutoff function, we estimate

$$\begin{aligned} \int_{\Gamma} [\nabla_{\tau} (\Delta u_n)]^2 ds &\leq O(\lambda_n^{1/2}) \left(\int_{\Gamma} \left[\frac{\partial}{\partial \nu} (\Delta u_n) \right]^2 ds \right)^{1/2} + O(\lambda_n^{3/2}) + \\ &+ O \left(\int_g [\nabla (\Delta u_n)]^2 dx \right). \end{aligned}$$

Taking into account that

$$\int_g [\nabla (\Delta u_n)]^2 dx = -\lambda_n \int_g u_n \Delta u_n dx + \int_{\Gamma} \Delta u_n \frac{\partial (\Delta u_n)}{\partial \nu} ds$$

and using the first estimate (5), we find

$$\int_g [\nabla (\Delta u_n)]^2 dx = O(\lambda_n^{3/2}), \quad \int_g \sum_{i,k,l=1}^m \left(\frac{\partial^3 u_n}{\partial x_i \partial x_k \partial x_l} \right)^2 dx = O(\lambda_n^{3/2}), \quad (8)$$

and then

$$\int_g \left[\frac{\partial}{\partial \nu} (\Delta u_n) \right]^2 ds = O(\lambda_n^{3/2}), \quad \int_{\Gamma} [\nabla_{\tau} (\Delta u_n)]^2 ds = O(\lambda_n^{3/2}). \quad (9)$$

We note that all estimates (4), (5), (8), and (9) are sharp.

Let $\delta, \delta \in (0, \delta_0)$, be sufficiently small. Denote by Γ_δ the equidistant surface at distance δ from Γ and lying inside the domain g , and by g^δ the domain bounded by Γ_δ .

Lemma 3. *There exist positive numbers δ, n_0, C_1, C_2 such that, if some eigenfunction $u_n(x)$ of problem (1) with number $n \geq n_0$ attains its maximum value at a point $x_0 \in (g - g^{\delta/2})$, then in the ball $K(x_0, \chi) \subset (g - g^\delta)$ with center at x_0 and radius $\chi = C_1 \lambda_n^{-1/4}$ there are two points x_1 and x_2 , lying on one normal to Γ , such that*

$$|u_n(x_1) - u_n(x_2)| \geq \frac{1}{C_2} |u_n(x_0)|.$$

This lemma is not difficult to derive from the mean-value theorem for the equation

$$\Delta^2 u - \lambda u = 0.$$

Theorem 1. *For the maximum of the modulus of the eigenfunctions of problem (1), the estimate (3) holds uniformly in the closed domain $(g + \Gamma)$.*

Proof. By the Green-Stokes formula, for any eigenfunction $u_n(x)$ we have the equality

$$u_n(x) = \int_{\Gamma} \left[G(x, y) \frac{\partial(\Delta u_n)}{\partial \nu} - \Delta u_n \frac{\partial G(x, y)}{\partial \nu} \right] ds_y; \quad (10)$$

here $G(x, y)$ is the fundamental solution of the equation $\Delta^2 v - \lambda v = 0$, equal to*

$$C r_{xy}^{(2-m)/2} \lambda_n^{(m-6)/8} \left[\frac{\pi}{2} N_{(m-2)/2} (r_{xy} \lambda_n^{1/4}) + K_{(m-2)/2} (r_{xy} \lambda_n^{1/4}) \right].$$

1°. Let $x \in \bar{g}_{\delta/2}$. Since $r_{x\Gamma} \geq \delta/2$, and since for the functions $N_\mu(z)$ and $K_\mu(z)$, for $z \geq 1$, the estimates $|N_\mu(z)| = O(1/\sqrt{z})$ and $|K_\mu(z)| = O((1/\sqrt{z})e^{-z})$ hold, from formula (10) we obtain

$$|u_n(x)| = O \left(\lambda_n^{(m-7)/8} \int_{\Gamma} \left| \frac{\partial(\Delta u_n)}{\partial \nu} \right| ds + \lambda_n^{(m-5)/8} \int_{\Gamma} |\Delta u_n| ds \right).$$

Hence, and from Lemma 2, the estimate follows

$$\max_{x \in \bar{g}_{\delta/2}} |u_n(x)| = O \left(\lambda_n^{(m-1)/8} \right). \quad (11)$$

2°. Let now $x \in \overline{(g - g_{\delta/2})}$. Differentiate (10) in the direction of the normal to Γ . Taking into account the formula $d(z^{-\mu} N_{\mu}(z))/dz = -z^{-\mu} N_{\mu+1}(z)$ and the analogous formula for $K_{\mu}(z)$, in the right-hand side of (10), after differentiation, we obtain four very similar terms. Let us consider one of them, for example,

$$I(x) = C \lambda_n^{(m-4)/8} \int_{\Gamma} r_{xy}^{(2-m)/2} N_{m/2}(r_{xy} \lambda_n^{1/4}) \cos(r_{xy}, \nu_x) \frac{\partial(\Delta u_n)}{\partial \nu} ds_y. \quad (12)$$

Denote by $z \in \Gamma$ the base of that normal to Γ on which the point x lies, and by ρ the distance from x to the point z : $\rho = r_{xz} = r_{x\Gamma}$. Split Γ into three parts: Γ_1 , the part of Γ lying outside the ball $K(z, R)$ with center at the point z and radius R , where $R > 0$ is a fixed number such that $\Gamma \cap K(z, R)$ admits, with respect to a local coordinate system $\xi_1, \dots, \xi_m$ with origin at z , a representation of the form $\xi_m = \zeta(\xi_1, \dots, \xi_{m-1})$; Γ_3 , the part of Γ lying inside the ball $K(z, \rho)$; $\Gamma_2 = \Gamma - (\Gamma_1 + \Gamma_3)$. Estimate the integral in (12) separately over each part Γ_i , assuming that $R_0 > R > \delta/2$, where R_0 is the radius of the Lyapunov sphere; we shall also assume that $\rho = r_{x\Gamma} \geq C \lambda_n^{-1/4}$. Then for $N_{m/2}(r_{xy} \lambda_n^{1/4})$ the estimate

$$|N_{m/2}(r_{xy} \lambda_n^{1/4})| = O\left(\frac{1}{r_{xy}^{1/2} \lambda_n^{1/8}}\right)$$

is valid,

$$\begin{aligned} \left| \int_{\Gamma_1} N_{m/2}(r \lambda_n^{1/4}) r^{(2-m)/2} \cos(r, \nu_x) \frac{\partial(\Delta u_n)}{\partial \nu} ds \right| &\leq C \int_{\Gamma_1} \frac{1}{r^{1/2} \lambda_n^{1/8}} r^{(2-m)/2} \left| \frac{\partial(\Delta u_n)}{\partial \nu} \right| ds \leq \\ &\leq C \lambda_n^{-1/8} \frac{1}{R^{(m-1)/2}} \left\{ \int_{\Gamma_1} \left[\frac{\partial(\Delta u_n)}{\partial \nu} \right]^2 ds \cdot \int_{\Gamma_1} ds \right\}^{1/2} \leq C \lambda_n^{5/8} \frac{1}{R^{(m-1)/2}} \leq C \lambda_n^{5/8}; \end{aligned} \quad (13)$$

$$\begin{aligned} \left| \int_{\Gamma_3} N_{m/2}(r \lambda_n^{1/4}) r^{(2-m)/2} \cos(r, \nu_x) \frac{\partial(\Delta u_n)}{\partial \nu} ds \right| &\leq C \int_{\Gamma_3} \frac{1}{r^{1/2} \lambda_n^{1/8}} r^{(2-m)/2} \left| \frac{\partial(\Delta u_n)}{\partial \nu} \right| ds \leq \\ &\leq C \lambda_n^{-1/8} \frac{1}{\rho^{(m-1)/2}} \left\{ \int_{\Gamma_3} \left[\frac{\partial(\Delta u_n)}{\partial \nu} \right]^2 ds \cdot \int_{\Gamma_3} ds \right\}^{1/2} \leq C \lambda_n^{5/8} \frac{1}{\rho^{(m-1)/2}} \rho^{(m-1)/2} = C \lambda_n^{5/8}. \end{aligned} \quad (14)$$

* All constants entering the formulas and independent of the number of the eigenfunctions are denoted by the

In the estimates (13) and (14) we have used (5) and the fact that for $y \in \Gamma_1$, $r_{xy} \geq R$, while for $y \in \Gamma_3$, $r_{xy} \geq \rho/2$.

Let τ denote the polar radius in the hyperplane tangent to Γ at the point z . Since $(g + \Gamma) \in A^{(4)}$, for $y \in \Gamma \cap K(z, R)$, $|\xi_m| = |\xi(\xi_1, \dots, \xi_{m-1})| \leq C\tau^2$. Obviously,

$$|\cos(r, \nu_x)| \leq (\rho + |\xi_m|)/\tau \leq \rho/\tau + C\tau.$$

Taking this into account, we estimate the integral over Γ_2 :

$$\begin{aligned} \left| \int_{\Gamma_2} N_{m/2}(r\lambda_n^{1/4}) r^{(2-m)/2} \cos(r, \nu_x) \frac{\partial(\Delta u_n)}{\partial \nu} ds \right| &\leq C\lambda_n^{-1/8} \int_{\Gamma_2} \left| \frac{\partial(\Delta u_n)}{\partial \nu} \right| r^{(1-m)/2} \left(\frac{\rho}{\tau} + C\tau \right) ds \\ &\leq C\lambda_n^{-1/8} \left\{ \int_{\Gamma_2} \left[\frac{\partial(\Delta u_n)}{\partial \nu} \right]^2 ds \cdot 2 \left[\int_{\Gamma'_2} \tau^{1-m} \frac{\rho^2}{\tau^2} ds' + \int_{\Gamma'_2} \tau^{1-m} C^2 \tau^2 ds' \right] \right. \\ &\quad \left. \leq C\lambda_n^{1/8} \left\{ \rho^2 \int_{\rho}^R \tau^{-m-1} \tau^{m-2} d\tau + \int_{\rho}^R \tau^{3-m} \tau^{m-2} d\tau \right\}^{1/2} \leq C\lambda_n^{5/8} \right\} \end{aligned} \quad (15)$$

the primes in the second inequality denote passage to integration over the hyperplane tangent at the point z .

Comparing (12), (13), (14), and (15), we obtain

$$|I(x)| = O\left(\lambda_n^{(m+1)/8}\right).$$

Since the remaining three terms obtained by differentiating formula (10) are estimated similarly, we finally find

$$|\partial u_n(x)/\partial \nu| = O\left(\lambda_n^{(N+1)/8}\right), \quad x \in (g - g_{\delta/2}), \quad \rho \geq C\lambda_n^{-1/4}. \quad (16)$$

Let x_0 be the point at which $|u(x)|$ attains its maximum, $U_0 = |u_n(x_0)|$; let z be the point at which $|\partial u_n(x)/\partial \nu|$ attains its maximum, $U_1 = |\partial u_n(z)/\partial \nu|$; and let $y_1(y_2) \in \Gamma$ be the foot of the normal to Γ on which $x_0(z)$ lies. a) If

$$r_{x_0\Gamma} \geq 2\chi = 2C_1\lambda_n^{-1/4},$$

then, applying Lagrange's formula to the two points mentioned in Lemma 3 and taking (16) into account, we obtain estimate (3). In (16), in this case, C must be taken to be C_1 , i.e. $\rho \geq C_1\lambda_n^{-1/4}$. b) If

$$r_{x_0\Gamma} < 2\chi = 2C_1\lambda_n^{-1/4},$$

then

$$U_0 = |u_n(x_0) - u_n(y_1)| = |\partial u(\xi)/\partial \nu| r_{x_0\Gamma},$$

i.e. $U_0 \leq U_1 r_{x_0\Gamma}$. Therefore

$$U_1 = |\partial u_n(z)/\partial \nu - \partial u_n(y_2)/\partial \nu| = |\partial^2 u_n(\eta)/\partial \nu^2| r_{z\Gamma} \leq C_0 \lambda_n^{1/2} U_0 r_{z\Gamma} \leq C_0 \lambda_n^{1/2} r_{z\Gamma} r_{x_0\Gamma} U_1$$

(the penultimate inequality follows from Schauder's estimate for problem (1)).

Hence

$$r_{z\Gamma} \geq (C_0 \lambda_n^{1/2} r_{x_0\Gamma})^{-1} > (2C_1 C_0 \lambda_n^{1/4})^{-1},$$

i.e. $\max_{x \in (g - g_{\delta/2})} |\partial u(x)/\partial \nu|$ is attained at a point z such that

$$r_{z\Gamma} > (2C_1 C_0)^{-1} \lambda_n^{-1/4}.$$

Therefore, in (16), C must be taken to be $(2C_1 C_0)^{-1}$. Estimate (3) is obtained by applying Lagrange's formula to the points x_0 and y_1 , taking into account (16) and the fact that

$$r_{x_0 y_1} < 2C_1 \lambda_n^{-1/4}.$$

Theorem 2. For the derivatives of the eigenfunctions of problem (1), the following estimate holds uniformly in the closed domain $(g + \Gamma)$:

$$\max_{x \in (g + \Gamma)} |D^k u_n(x)| = O\left(\lambda_n^{(m-1)/8+k/4}\right), \quad k = 0, 1, 2, \dots \quad (17)$$

Estimate (17) is sharp. Theorem 2 follows easily from Theorem (1), Schauder's estimates, and interpolation inequalities (see ⁴).

Remark. Estimates (3) and (17) are also valid in the case when, in problem (1), the boundary conditions are taken to be

$$\Delta u|_{\Gamma} = 0, \quad \partial(\Delta u)/\partial \nu|_{\Gamma} = 0.$$

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