

ON THE APPROXIMATION OF A FUNCTION BY RATIONAL FRACTIONS

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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF A FUNCTION BY RATIONAL FRACTIONS

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In this note conditions are formulated on a function under which it can be represented as the limit of a uniformly convergent sequence of rational fractions on a certain prescribed set.

We denote by τ the plane of the complex variable $z = x + iy$; by Ce the complement of the set e in τ ; by $A(e, m)$ the set of all functions each of which is analytic outside some closed subset of the set e , is bounded there in modulus by the constant m , and is equal to zero at infinity;

$\gamma(e, f) = \lim_{z \rightarrow \infty} z f(z)$, where $f(z) \in A(e, m)$;

$\gamma(e) = \sup_{f \in A(e, 1)} |\gamma(e, f)|$ is the analytic capacity of the set e ;

$\varphi(z, e)$ is the Ahlfors function for the closed set e , i.e. $\varphi(z, e) \in A(e, 1)$ and $\gamma(e, \varphi) = \gamma(e)$;

$K(z, \delta)$ is the square whose sides are parallel to the coordinate axes, with center at z and side length δ ; ∂K is the boundary of the set K .

Theorem. Let e be a closed bounded set; let $f(z)$ be a function continuous in the whole plane and such that, for every square $K(z, \delta)$,

$$\left| \int_{\partial K(z, \delta)} f(\zeta) d\zeta \right| \leq \gamma[Ce \cap K(z, r\delta)] \Omega(\delta),$$

where $r \geq 1$ does not depend on z and δ ; $\lim_{\delta \rightarrow 0} \Omega(\delta) = 0$. Then $f(z)$ can be uniformly approximated with arbitrary accuracy on the set e by a rational fraction*.

Lemma 1. Let α be a natural number; $c_{k,n} \leq c$ ($k = 1, 2, \dots$; $n = 1, 2, \dots, k\alpha$) be nonnegative numbers. Then

$$\sum_{k,n} c_{k,n} \geq B(\alpha) c \left[\sum_{k,n} \frac{c_{k,n}}{kc} \right]^2,$$

where $B(\alpha)$ depends only on α .

Lemma 2. Let $\{e_i\}$ be a system of closed subsets of e such that, for every square $K(z, \gamma(e))$, the number of sets of this system intersecting it is not greater than p . Then for all functions $\{f_i(z) \in A(e_i, 1)\}$ the inequality

$$\sup_{z \in Ce} \sum_i |f_i(z)| \leq c(p)$$

holds.

Proof. Fix a point ζ and renumber $\{e_i\}$ with two indices k, n so that $k = [\rho^{-1}(\zeta, e_{k,n})\gamma(e)]$, where $\rho(\zeta, e_{k,n})$ is the distance from the point ζ to $e_{k,n}$. We obtain that the index n runs through no more

* *Remark in proof.* The condition of the theorem is also a necessary condition for approximability of a function by rational fractions. The proof of this fact has been submitted for publication.

$ak = a(p)k$ values. Put

$$\mu(\zeta) = \sum_{k,n} \frac{\gamma(e_{k,n})}{k\gamma(e)}; \quad \mu = \sup_{\zeta} \mu(\zeta); \quad \varphi(z) = \sum_i \varphi(z, e_i),$$

Since for every $g(z) \in A(e, 1)$, $|g(z)| \leq \min\{1, \gamma(e)/\rho(z, e)\}$, it follows that

$$\sup_z |\varphi(z)| \leq C_1(p) \left[1 + \sum_{k,n} \frac{\gamma(e_{k,n})}{k\gamma(e)} \right] \leq C_1(p)(1 + \mu) = \mu_1.$$

Since $\sum_i \gamma(e_i) \leq \mu_1 \gamma(e)$ and since, by Lemma 1, $\sum_i \gamma(e_i) \geq B(\alpha) \gamma(e) \mu^2$, we have $B(\alpha) \mu^2 \leq \mu_1$, i.e. $\mu \leq C_2(p)$. Therefore

$$\sum_i |f_i(\zeta)| \leq C_1(p)(1 + \mu) = C(p).$$

The lemma is proved.

Lemma 3. If $\{e_i\}$ satisfies the conditions of Lemma 2, then

$$\sum_i \gamma(e_i) \leq C(p) \gamma(e).$$

Lemma 4. If $f(z)$ satisfies the conditions of the theorem, then for all z and δ

$$\left| \int_{\partial K(z, \delta)} f(\zeta)(\zeta - z) d\zeta \right| \leq \delta \gamma(Ce \cap K(z, \delta)) \Phi(\delta),$$

and $\Phi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. Fix a natural number q and consider the system of squares $K_\beta^s = K(z_\beta^s, \delta q^{-s})$ ($\beta = (\beta_0, \dots, \beta_s)$; $\beta_i = 1, \dots, q^2$; $s = 0, 1, 2, \dots$) such that $K_\beta^0 = K(z, \delta)$, for every s any two squares from $\{K_\beta^s\}$ are congruent, and for all s and β

$$K_\beta^s = \bigcup_{\beta_{s+1}=1}^{q^2} K_\beta^{s+1}.$$

We shall denote by the sign $\sum'_\beta$ (respectively $\sum''_\beta$) summation over all those values of β for which $\gamma(Ce \cap K_\beta^s) \geq q^{-1} \delta q^{-s}$ (respectively summation over all remaining values of β). For all s, β ,

$$\begin{aligned} \int_{\partial K_\beta^{s-1}} f(\zeta)(\zeta - z_\beta^{s-1}) d\zeta &= \sum_{\beta_s} (z_\beta^s - z_\beta^{s-1}) \int_{\partial K_\beta^s} f(\zeta) d\zeta + \\ &+ \sum'_{\beta_s} \int_{\partial K_\beta^s} f(\zeta)(\zeta - z_\beta^s) d\zeta + \sum''_{\beta_s} \int_{\partial K_\beta^s} f(\zeta)(\zeta - z_\beta^s) d\zeta; \end{aligned}$$

therefore, for every n ,

$$\begin{aligned} \int_{\partial K_\beta^0} f(\zeta)(\zeta - z) d\zeta &= \sum_{s=1}^{n-1} \sum''_{\beta_0, \dots, \beta_{s-1}} \left\{ \sum_{\beta_s} (z_\beta^s - z_\beta^{s-1}) \int_{\partial K_\beta^s} f(\zeta) d\zeta + \right. \\ &\left. + \sum'_{\beta_s} \int_{\partial K_\beta^s} f(\zeta)(\zeta - z_\beta^s) d\zeta \right\} + \sum''_{\beta_0, \dots, \beta_n} \int_{\partial K_\beta^n} f(\zeta)(\zeta - z_\beta^n) d\zeta. \end{aligned}$$

Denote by $\omega(\delta)$ the modulus of continuity of $f(z)$ and put

$$\psi(\delta) = \max_{t \leq \delta} [\Omega(t) + \omega(t)]$$

and

$$e_\beta^s = Ce \cap K(z_\beta^s, r^0 \delta q^{-s}).$$

From the inequality

$$\left| \int_{\partial K_\beta^s} f(\zeta)(\zeta - z_\beta^s) d\zeta \right| \leq \omega(\delta q^{-s}) \cdot 4\delta q^{-s}$$

and Lemma 2 we obtain

$$\begin{aligned} \left| \int_{\partial K(z, \delta)} f(\zeta)(\zeta - z) d\zeta \right| &\leq \sum_{s=1}^{n-1} \sum''_{\beta_0, \dots, \beta_{s-1}} \left\{ \sum_{\beta_s} |z_\beta^s - z_\beta^{s-1}| \gamma(e_\beta^s) \Omega(\delta q^{-s}) + \right. \\ &\quad \left. + \sum'_{\beta_s} \omega(\delta q^{-s}) \delta q^{-s} \cdot 4\delta q^{-s} + \sum''_{\beta_0, \dots, \beta_n} \omega(\delta q^{-n}) \cdot 4(\delta q^{-n})^2 \leq \right. \\ &\leq \sum_{s=1}^{n-1} \sum''_{\beta_0, \dots, \beta_{s-1}} \left\{ \delta q^{1-s} \psi(\delta) \sum_{\beta_s} \gamma(e_\beta^s) + 4\delta q^{-s} \psi(\delta) \sum'_{\beta_s} q \gamma(Ce \cap K_\beta^s) \right\} \\ &\quad + 4\delta^2 \omega(\delta q^{-n}) \leq 4\delta \psi(\delta) \sum_{s=1}^{\infty} q^{1-s} \sum''_{\beta_0, \dots, \beta_{s-1}} \{c(p)[\gamma(e_\beta^{s-1}) + \gamma(Ce \cap K_\beta^{s-1})]\} \leq \\ &\leq 8\delta \psi(\delta) \sum_{s=1}^{\infty} q^{1-s} [c(p)]^s \gamma(e_\beta^0), \end{aligned}$$

whence, for $q = [2c(p) + 2]$, $\Phi(\delta) = 16\psi(\delta)$, we obtain the assertion of the lemma.

Denote:

$$\beta(e, z, f) = [\gamma(e)2\pi i]^{-1} \int_{\partial e} f(\zeta)(\zeta - z) d\zeta; \quad \beta(e, z) = \sup_{f \in A(e, 1)} |\beta(e, z, f)|;$$

$$\beta(e) = \min_z \beta(e, z)$$

—the “analytic diameter” of the set e ; $O(e)$ is the “center,” i.e., a point such that $\beta(e, O(e)) = \beta(e)$.

Lemma 5. If e and $f(z)$ satisfy the conditions of the theorem, then for every square $K(z, \delta)$

$$\left| \int_{\partial K(z, \delta)} f(\zeta)[\zeta - O(Ce \cap K(z, r\delta))] d\zeta \right| \leq$$

$$\leq \Lambda(\delta) \gamma(Ce \cap K(z, r\delta)) \beta(Ce \cap K(z, r\delta))$$

and $\Lambda(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Proof. Divide $K(z, \delta)$ into equal squares with side β and centers at the points t_i . Put $e(z, \delta) = Ce \cap K(z, r\delta)$. Choose β so that $\beta \leq \beta(e(z, \delta)) \leq 2\beta$, if $\beta(e(z, \delta)) \leq \delta$, and $\beta = \delta$, if $\beta(e(z, \delta)) > \delta$. From the condition of the theorem and Lemmas 4 and 3 we obtain

$$\begin{aligned} & \left| \int_{\partial K(z, \delta)} f(\zeta) [\zeta - O(e(z, \delta))] d\xi \right| \leq \\ & \leq \sum_i |t_i - O(e(z, \delta))| \gamma(e(t_i, \beta)) \Omega(\beta) + \sum_i \gamma(e(t_i, \beta)) \beta \Phi(\beta) \leq \\ & \leq \sum_i |t_i - O(e(z, \delta))| \gamma(e(t_i, \beta)) \Omega(\beta) + c(p) \gamma(e(z, \delta)) \beta \Phi(\beta). \end{aligned}$$

Let $f_i = f_i(z) \in A(e(t_i, \beta), 1)$ and

$$\gamma(e(t_i, \beta), f_i) = \frac{1}{2} [t_i - O(e(z, \delta))]^{-1} |t_i - O(e(z, \delta))| \gamma(e(t_i, \beta));$$

then $\varphi(z) = \sum_i f_i(z)$ is bounded in modulus by $c(p)$ (see Lemma 2). Consequently,

$$\beta(e(z, \delta)) \geq |(c(p))^{-1} \beta(e(z, \delta)), O(e(z, \delta))|,$$

$$\begin{aligned} |\varphi(z)| & \geq [2\pi \gamma(e(z, \delta)) c(p)]^{-1} \left| \int \left(\sum_i [t_i - O(e(z, \delta))] f_i(\zeta) + (\zeta - t_i) f_i(\zeta) \right) d\xi \right| \geq \\ & \geq [2\pi \gamma(e(z, \delta)) c(p)]^{-1} \left\{ \sum_i \frac{1}{2} |t_i - O(e(z, \delta))| \gamma(e(t_i, \beta)) - m_1(r) \beta \gamma(e(z, \delta)) \right\}, \end{aligned}$$

where $m_1(r)$ is a constant.

Thus,

$$\left| \int_{\partial K(z, \delta)} f(\zeta) [\zeta - O(e(z, \delta))] d\zeta \right| \leq$$

$$\begin{aligned} &\leq \sum_i |t_i - O(e(z, \delta))| \gamma(e(t_i, \beta)) \Omega(\beta) + c(p) \gamma(e(z, \delta)) \beta \Phi(\beta) \leq \\ &\leq \gamma(e(z, \delta)) \beta(e(z, \delta)) m_2(r) [\Omega(\beta) + \Phi(\beta)]. \end{aligned}$$

The lemma is proved.

Lemma 6. If the complex numbers γ and β are such that $|\gamma| \leq \gamma(e)$ and $|\beta| \leq \beta(e)$, then there exists a function $g(z) \in A(e, 5)$ for which $\gamma(e, g) = \gamma$ and $\beta(e, O(e), g) = \beta$.

Proof. For every function $f(z)$,

$$\beta(e, \zeta, f) = \beta(e, O(e), f) + \frac{\gamma(e, f)}{\gamma(e)} (\zeta - O(e)).$$

Consequently, for $\varphi = \varphi(z, e)$, when

$$z_0 = O(e) - \beta(e, O(e), \varphi)$$

we have $\beta(e, z_0, \varphi) = 0$.

From the definition of $O(e)$ it follows that for some function $f_1 \in A(e, 1)$,

$$\beta(e, z_0, f_1) = \beta(e).$$

Choose ε so that for

$$f_2 = (f_1 + \varepsilon \varphi) \in A(e, 2)$$

one has $\beta(e, O(e), f_2) = \beta(e)$ and $\gamma(e, f_2) = 0$. Since

$$\beta(e, O(e), \varphi) = O(e) - z_0,$$

we have $|O(e) - z_0| \leq \beta(e)$. Then

$$f = \frac{\gamma}{\gamma(e)} \varphi - \frac{\gamma[O(e) - z_0]}{\gamma(e)\beta(e)} f_2 + \frac{\beta}{\beta(e)} f_2 \in A(e, 5)$$

is the desired function.

Proof of the theorem. Let K be a square containing the set e . Divide it into equal squares

$$K_i = K(t_i, \delta), \quad O_i = O(Ce \cap K(t_i, r\delta)).$$

Let

$$f^* = \frac{1}{2\pi i} \int_{\partial K} \frac{f^*(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int_K \frac{df^*}{d\bar{\zeta}} \frac{1}{\zeta - z} dS$$

be a smooth function uniformly approximating f on K with accuracy up to ε . By Lemma 6, there exist functions

$$g_i \in A(Ce, 5(\Delta(\delta) + \Omega(\delta))),$$

whose first two coefficients in the expansion in powers of $1/z - O_i$ are equal to the corresponding coefficients of the function

$$f_i^* = \frac{1}{\pi} \int_{K_i} \frac{df^*}{d\zeta} \frac{1}{\zeta - z} dS.$$

It remains to verify that, if one first chooses sufficiently small δ , and then ε , then the function analytic in a neighborhood of the set e ,

$$g(z) = \frac{1}{2\pi i} \int_{\partial K} \frac{f^*(\zeta)}{\zeta - z} d\zeta + \sum g_i$$

will be sufficiently close to $f(z)$. Approximating this function by a rational fraction, we obtain the assertion of the theorem.

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Note: Figure translations are in progress. See original paper for figures.

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