

EXPANSION IN EIGENFUNCTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

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Abstract

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MATHEMATICS

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EXPANSION IN EIGENFUNCTIONS OF A ONE-DIMENSIONAL DIRAC SYSTEM

(Presented by Academician I. N. Vekua on 27 V 1965)

1. Let $\mathcal{L}$ denote the matrix operator

$$\mathcal{L} \equiv \begin{pmatrix} p(x) & d/dx \\ -d/dx & r(x) \end{pmatrix},$$

where $p(x)$ and $r(x)$ are real-valued functions defined on the half-line $(0, \infty)$ and summable on every finite interval. Further, let $y(x)$ denote a two-component vector-function

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}.$$

Then the equation (λ is a parameter)

$$(\mathcal{L} - \lambda)y = 0$$

is equivalent to the system of two compatible ordinary differential equations of the first order

$$dy_2/dx + p(x)y_1 = \lambda y_1, \quad (1)$$

$$-dy_1/dx + r(x)y_2 = \lambda y_2. \quad (2)$$

The system (1)–(2) is a one-dimensional analogue of the stationary system in the relativistic quantum theory of Dirac ⁽¹⁾

$$(W - V + mc^2)X_1 - ic \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) X_4 - ic \frac{\partial X_3}{\partial z} = 0,$$

$$\begin{aligned}
(W - V + mc^2)X_2 - ic \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) X_3 + ic \frac{\partial X_4}{\partial z} &= 0, \\
(W - V - mc^2)X_3 - ic \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) X_2 - ic \frac{\partial X_1}{\partial z} &= 0, \\
(W - V - mc^2)X_4 - ic \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) X_1 + ic \frac{\partial X_2}{\partial z} &= 0,
\end{aligned} \tag{3}$$

where $V = V(x, y, z)$ is the potential function, c is the speed of light, W is the energy, and m is the mass of the particle.

Indeed, if $X_1(x, y, z) = X_4(x, y, z) \equiv 0$, and the functions $V(x, y, z)$, $X_2(x, y, z)$, and $X_3(x, y, z)$ do not depend on x and z , then system (3) takes the form

$$\begin{aligned}
\{W - V(y) + mc^2\}X_2 + c \frac{dX_3}{dx} &= 0, \\
\{W - V(y) - mc^2\}X_3 - c \frac{dX_2}{dx} &= 0.
\end{aligned} \tag{4}$$

Let the units of measurement be chosen so that the speed of light $c = 1$. If we put $p(y) = V(y) + m$, $r(y) = V(y) - m$, $\lambda = W$, then systems (1)–(2) and (4) will differ only in notation.

We adjoin to system (1)–(2) the boundary condition

$$y_2(0) - hy_1(0) = 0, \tag{5}$$

where h is an arbitrary real number.

Consider the eigenvalue problem (1) + (2) + (5).

In the present note, by a method analogous to the method of B. M. Levitan (3), we study the question of expanding an arbitrary vector-function $f(x) = \{f_1(x), f_2(x)\}$ with square integrable norm on the half-line $(0, \infty)$,

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\} dx < +\infty,$$

in the eigenfunctions of the problem (1) + (2) + (5).

2. Denote by $\varphi(x, \lambda) = \{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ the solution of the system (1)–(2) satisfying the initial conditions

$$\varphi_1(0, \lambda) = 1, \quad \varphi_2(0, \lambda) = h. \quad (5')$$

It is obvious that the vector-function $\varphi(x, \lambda)$ satisfies condition (5).

It is known (2) that, for a given h , to each boundary-value problem (1) + (2) + (5) there corresponds a unique nondecreasing, bounded on every finite interval, left-continuous function $\rho(\lambda)$ ($-\infty < \lambda < \infty$), generating an isometric mapping of the space of vector-functions $f(x) \in L^2(0, \infty)$ onto the space $L^2_{\{\rho(\lambda)\}}(-\infty, \infty)$ by the formulas

$$F(\lambda) = \int_0^\infty \{f_1(x)\varphi_1(x, \lambda) + f_2(x)\varphi_2(x, \lambda)\} dx, \quad (6)$$

$$f_1(x) = \int_{-\infty}^\infty F(\lambda)\varphi_1(x, \lambda) d\rho(\lambda), \quad (7)$$

$$f_2(x) = \int_{-\infty}^\infty F(\lambda)\varphi_2(x, \lambda) d\rho(\lambda), \quad (8)$$

where the integrals (6) and (7)–(8) converge respectively in the metrics of the spaces $L^2_{\{\rho(\lambda)\}}(-\infty, \infty)$ and $L^2(0, \infty)$, and Parseval's equality holds:

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\} dx = \int_{-\infty}^\infty F^2(\lambda) d\rho(\lambda).$$

Put $(i, k = 1, 2)$

$$\Theta_{ik}(x, s; \lambda) = \begin{cases} \int_0^\lambda \varphi_i(x, \lambda)\varphi_k(s, \lambda) d\rho(\lambda), & \lambda > 0, \\ -\int_\lambda^0 \varphi_i(x, \lambda)\varphi_k(s, \lambda) d\rho(\lambda), & \lambda < 0, \\ 0, & \lambda = 0. \end{cases}$$

The square matrix of second order $\Theta(x, s; \lambda) = \{\Theta_{ik}(x, s; \lambda)\}$ is called the spectral matrix of the boundary-value problem (1) + (2) + (5).

Let the vector-function $f(x) = \{f_1(x), f_2(x)\} \in L^2(0, \infty)$, i.e.

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\} dx < +\infty.$$

Introduce the notation

$$S_1(x, \lambda) = \int_0^\infty \{f_1(s)\Theta_{11}(x, s; \lambda) + f_2(s)\Theta_{12}(x, s; \lambda)\} ds, \quad (9)$$

$$S_2(x, \lambda) = \int_0^\infty \{f_1(s)\Theta_{21}(x, s; \lambda) + f_2(s)\Theta_{22}(x, s; \lambda)\} ds. \quad (10)$$

By virtue of the definition of the functions $\Theta_{ik}(x, s; \lambda)$ and equality (6), for the functions $S_1(x, \lambda)$ and $S_2(x, \lambda)$ we obtain the expressions

$$S_1(x, \lambda) = \int_0^\lambda F(\lambda)\varphi_1(x, \lambda) d\rho(\lambda),$$

$$S_2(x, \lambda) = \int_0^\lambda F(\lambda)\varphi_2(x, \lambda) d\rho(\lambda).$$

The functions $S_1(x, \lambda)$ and $S_2(x, \lambda)$ are segments of the expansions, respectively, of the functions $f_1(x)$ and $f_2(x)$ in the Fourier integral with respect to the eigenfunctions of problem (1) + (2) + (5).

Consider problem (1) + (2) + (5') for $p(x) = r(x) = 0$ and $h = 0$. Then $\varphi_1(x, \lambda) = \cos \lambda x$, $\varphi_2(x, \lambda) = \sin \lambda x$. Therefore, in the case under consideration the spectral matrix, which we denote by $\Theta^*(x, s; \lambda)$, is determined by the formulas

$$\Theta_{11}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \cos \lambda x \cos \lambda s d\lambda,$$

$$\Theta_{12}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \cos \lambda x \sin \lambda s d\lambda,$$

$$\Theta_{21}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \sin \lambda x \cos \lambda s d\lambda,$$

$$\Theta_{22}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \sin \lambda x \sin \lambda s d\lambda.$$

Consequently, the functions (analogously to the preceding)

$$S_1^*(x, \lambda) = \int_0^\infty \{f_1(s)\Theta_{11}^*(x, s; \lambda) + f_2(s)\Theta_{12}^*(x, s; \lambda)\} ds,$$

$$S_2^*(x, \lambda) = \int_0^\infty \{f_1(s)\Theta_{21}^*(x, s; \lambda) + f_2(s)\Theta_{22}^*(x, s; \lambda)\} ds$$

are segments of the expansions of the functions $f_1(s)$ and $f_2(s)$ in the ordinary Fourier integral.

Using the asymptotic estimates for the spectral matrix

$$\Theta(x, s; \lambda) = \{\Theta_{ik}(x, s; \lambda)\} \quad (i, k = 1, 2),$$

obtained in note ⁽⁵⁾, on the basis of the definition of the functions $S_1(x, \lambda)$ and $S_2(x, \lambda)$, i.e., equalities (9) and (10), the following is proved.

Lemma 1. If the coefficients $p(x)$ and $r(x)$ are summable on each finite interval, then for every fixed x and as $|a| \rightarrow \infty$ the asymptotic estimates

$$\bigvee_a^{a+1} \{S_i(x, \lambda)\} = o(1), \quad (i = 1, 2). \quad (11)$$

hold. The asymptotic estimates (11) hold uniformly in each finite interval of variation of x .

From the definitions of the spectral matrix

$$\Theta^*(x, s; \lambda) = \{\Theta_{ik}^*(x, s; \lambda)\}$$

and the functions $S_1^*(x, \lambda)$ and $S_2^*(x, \lambda)$ it follows directly that

Lemma 2. As $|a| \rightarrow \infty$, uniformly on the half-line $(0, \infty)$, the asymptotic estimate

$$\bigvee_a^{a+1} \{S_i^*(x, \lambda)\} = o(1) \quad (i = 1, 2)$$

holds.

Lemmas 1 and 2 make it possible to apply Tauberian theorem for Fourier integrals of B. M. Levitan ⁽⁴⁾.

As a result, one can obtain the following theorem:

Theorem 1 (on equiconvergence). Let $f(x) = \{f_1(x), f_2(x)\} \subset L^2(0, \infty)$. If the coefficients $p(x)$ and $r(x)$ are summable on every finite interval, then for each fixed x the equalities

$$\lim_{\lambda \rightarrow \infty} \{[S_1(x, \lambda) - S_1(x, -\lambda)] - [S_1^*(x, \lambda) - S_1^*(x, -\lambda)]\} = 0, \quad (12)$$

$$\lim_{\lambda \rightarrow \infty} \{[S_2(x, \lambda) - S_2(x, -\lambda)] - [S_2^*(x, \lambda) - S_2^*(x, -\lambda)]\} = 0. \quad (13)$$

hold.

The equalities (12) and (13) hold uniformly on every finite interval of variation of x .

Since, by virtue of the definitions of the functions $S_1(x, \lambda)$ and $S_2^*(x, \lambda)$,

$$S_1^*(x, \lambda) - S_1^*(x, -\lambda) = \frac{1}{\pi} \int_{-\lambda}^{\lambda} \left\{ \int_0^{\infty} f_1(s) \cos \lambda s ds \right\} \cos \lambda x d\lambda,$$

$$S_2^*(x, \lambda) - S_2^*(x, -\lambda) = \frac{1}{\pi} \int_{-\lambda}^{\lambda} \left\{ \int_0^{\infty} f_2(s) \sin \lambda s ds \right\} \sin \lambda x d\lambda,$$

the equalities (12) and (13) have the form

$$\lim_{\lambda \rightarrow \infty} \left\{ S_1(x, \lambda) - S_1(x, -\lambda) - \frac{1}{\pi} \int_{-\lambda}^{\lambda} \left[\int_0^{\infty} f_1(s) \cos \lambda s ds \right] \cos \lambda x d\lambda \right\} = 0, \quad (14)$$

$$\lim_{\lambda \rightarrow \infty} \left\{ S_2(x, \lambda) - S_2(x, -\lambda) - \frac{1}{\pi} \int_{-\lambda}^{\lambda} \left[\int_0^{\infty} f_2(s) \sin \lambda s ds \right] \sin \lambda x d\lambda \right\} = 0. \quad (15)$$

The equalities (14) and (15) mean that, uniformly on every interval of the half-line $(0, \infty)$, the difference between the expansion of an arbitrary vector-function $f(x) = \{f_1(x), f_2(x)\} \subset L^2(0, \infty)$ in the generalized Fourier integral with respect to the eigenfunctions of the one-dimensional Dirac operator and the expansion in the ordinary Fourier integral tends to zero.

Theorem 1 gives the final solution of the question of convergence of the expansion in eigenfunctions of the one-dimensional Dirac operator for a two-component vector-function with integrable square.

In particular, from Theorem 1 it follows that

Theorem 2 (on convergence). If the coefficients $p(x)$ and $r(x)$ are summable functions on every finite interval and the vector-function $f(x) = \{f_1(x), f_2(x)\}$ belongs to the class $L^2(0, \infty)$, then at every point where the local conditions of expandability of the vector-function $f(x)$ into the ordinary Fourier integral are satisfied, the equalities

$$\lim_{\lambda \rightarrow \infty} \{S_1(x, \lambda) - S_1(x, -\lambda)\} = f_1(x),$$

$$\lim_{\lambda \rightarrow \infty} \{S_2(x, \lambda) - S_2(x, -\lambda)\} = f_2(x),$$

hold, i.e. the expansion of the vector-function $f(x) = \{f_1(x), f_2(x)\}$ in the generalized Fourier integral with respect to the eigenfunctions of the one-dimensional Dirac operator tends to the value of the vector-function.

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REFERENCES

1. E. C. Titchmarsh, *Proc. London Math. Soc.*, (3), **11**, 169 (1961).
2. E. C. Titchmarsh, *Proc. London Math. Soc.*, (3), **11**, 159 (1961).
3. B. M. Levitan, *Izv. AN SSSR, seriya matem.*, **16**, 325 (1952).
4. B. M. Levitan, I. S. Sargsyan, *UMN*, **15**, no. 1 (91), 3 (1960).
5. I. S. Sargsyan, *DAN*, **166**, no. 5 (1966).

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