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Abstract

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MATHEMATICS

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GEOMETRY OF AN m -SURFACE IN AFFINE SPACE

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The geometry of an m -surface in affine space for $m = n - 1$ and $m = n - 2$ was studied by V. V. Wagner ^(1,9). A. E. Liber ⁽²⁾ considered the geometry of an m -surface under the condition that m and n satisfy the inequality $2n \leq m(m + 3)$. Later, A. E. Liber ⁽³⁻⁵⁾ and P. I. Shveikin ^(6,7) studied the geometry of an m -surface for arbitrary m , but under the condition that the tangent spaces of higher order have the greatest possible dimension. We impose no restrictions on m and n and do not require the greatest possible dimension of the tangent spaces of higher order, but assume only that all points of the surface are ordinary.

Let

$$x^\alpha = l^\alpha(\eta^a) \quad (\alpha, \beta, \gamma, \dots = 1, \dots, n; \ a, b, c, \dots = 1, \dots, m) \quad (1)$$

be the equations of an m -surface. We shall call an m -surface p -planar if it lies in some p -dimensional plane. Otherwise the surface will be called nonplanar. Since the study of a planar surface can be reduced to the study of a nonplanar surface in some affine space of smaller dimension, it is natural to restrict ourselves to considering the case of a nonplanar surface.

Let the tangent spaces of the m -surface up to order ν inclusive ($\nu \geq 1$) have the greatest possible dimension, but less than n , and let, beginning with order $(\nu + 1)$, they have dimension lower than the greatest possible. Then there exists an s such that the dimension of the tangent space spanned by the vectors

$$\partial_{a_1} \dots \partial_{a_i} l^\alpha \quad (i = 1, \dots, \nu + s), \quad (2)$$

is equal to n , while the dimension of the tangent space of order lower than $(\nu + s)$ is less than n . A surface of this type will be called a surface of differential class $(\nu + s)$.

If $N_\alpha^{p_s}$ ($p_s = 1, \dots, m_s$) is a solution of the system of equations

$$N_\alpha \partial_{a_1} \dots \partial_{a_i} l^\alpha = 0 \quad (i = 1, \dots, \nu + s - 1), \quad (3)$$

then the associated affnor

$$H_{a_1 \dots a_{\nu+s}}^{p_s} = N_\alpha^{p_s} \partial_{a_1} \dots \partial_{a_{\nu+s}} l^\alpha \quad (4)$$

is different from zero. Suppose that there exists a relative invariant $W^{(s)}$ of the associated affnor (4), not equal to zero. As is known ⁽³⁾, the weights k_1 and k_2 of the relative invariant $W^{(s)}$ are in the ratio $m_s(\nu + s) : (-m)$. Raising $W^{(s)}$ to a suitable power, we obtain a relative invariant $I^{(s)}$ with weights $k_1 = m_s(\nu + s)$ and $k_2 = -m$.

The associated affnors (4) and

$$H_{p_s}^{a_1 \dots a_{\nu+s}} = \frac{1}{I^{(s)}} \frac{\partial I}{\partial H_{a_1 \dots a_{\nu+s}}^{p_s}} \quad (5)$$

satisfy the conditions

$$H_{a_1 \dots a_{\nu+s-1} a}^{p_s} H_{p_s}^{a_1 \dots a_{\nu+s-1} b} = m_s \delta_a^b, \quad (6)$$

$$H_{a_1 \dots a_{\nu+s}}^{p_s} H_{q_s}^{a_1 \dots a_{\nu+s}} = m \delta_{q_s}^{p_s}, \quad (7)$$

and therefore the vectors

$$n_{p_s}^\alpha = \frac{1}{m} H_{p_s}^{a_1 \dots a_{\nu+s}} \partial_{a_1} \dots \partial_{a_{\nu+s}} l^\alpha \quad (8)$$

will be basis vectors of a certain vector space B_{m_s} .

If $N_\alpha^{p_{s-1}}$ ($p_{s-1} = 1, \dots, m_{s-1}$) is a solution of the system of equations

$$N_\alpha n_{p_s}^\alpha = 0; \quad N_\alpha \partial_{a_1} \dots \partial_{a_i} l^\alpha = 0 \quad (i = 1, \dots, \nu + s - 2), \quad (9)$$

then the associated affnor

$$H_{a_1 \dots a_{\nu+s-1}}^{p_{s-1}} = N_\alpha^{p_{s-1}} \partial_{a_1} \dots \partial_{a_{\nu+s-1}} l^\alpha \quad (10)$$

will be different from zero. If there exists a relative invariant $W^{(s-1)}$ of the affnor (10) that is not equal to zero, then, raising it to a suitable power, we

obtain a relative invariant I with weights $k_1 = m_{s-1}(\nu + s - 1)$ and $k_2 = -m$. The vectors

$$n_{p_{s-1}}^\alpha = \frac{1}{m} H_{p_{s-1}}^{a_1 \dots a_{\nu+s-1}} \partial_{a_1} \dots \partial_{a_{\nu+s-1}} l^\alpha \quad (11)$$

will be basis vectors of a certain vector space $B_{m_{s-1}}$. Similarly, one finds the basis vectors $n_{p_i}^\alpha$ ($i = 1, \dots, s - 2$) of the vector spaces B_{m_i} .

It is not difficult to see that

$$m_1 + m_2 + \dots + m_s = n + 1 - \binom{m + \nu}{m} \quad (12)$$

and the vectors

$$\partial_{a_1} \dots \partial_{a_i} l^\alpha; \quad n_{p_k}^\alpha \quad (i = 1, \dots, \nu; \quad k = 1, \dots, s) \quad (13)$$

are linearly independent. Therefore the vector spaces B_m^k ($k = 1, \dots, s$) partially equip the m -surface.

If the relative invariants W of the m -surface are not equal to zero, then the method proposed by the author [8] for constructing relative invariants of associated affinors of type (4) makes it possible to find the basis vectors $n_{p_i}^\alpha$ of the partially equipping spaces B_{m_i} . Thus, for example, for the two-dimensional surface in six-dimensional space

$$\bar{r}\{u; v; u^2; v^2; u^3; v^3\}, \quad (14)$$

which has differential class $(1 + 2)$, as the relative invariants $\overset{(2)}{W}$ and $\overset{(1)}{W}$ one may take the hyperdeterminant of the matrix

$$H^{p_2 q_2 s_2 r_2} = H_{1[1] \atop (2)(3)}^{\{p_2\}} \left[H_{1[1] \atop (4)(5)(6)}^{\{q_2\}} H_{2[2] \atop (2)(3)}^{\{s_2\}} \right] H_{2[2] \atop (4)(5)(6)}^{r_2}, \quad (15)$$

where symmetrization is carried out over the combined index $\{pq\}$, and alternation over each i -th index of adjacent elements separately ($i = 2, 3, 4, 5, 6$), and

$$\overset{(1)}{W} = \text{Det} (H^{p_1 q_1}), \quad (16)$$

where

$$H^{p_1 q_1} = H_{1[1]}^{(p_1)} H_{[2]2]}^{(q_1)}. \quad (17)$$

Simple computations show that the relative invariants $\overset{(1)}{W}$ and $\overset{(2)}{W}$ are not equal to zero, and therefore may be used for constructing partially normalizing spaces.

Since the vectors (13) form a basis, the vector $\partial_{a_1} \dots \partial_{a_{\nu+1}} l^\alpha$ admits a unique expansion in these vectors. Let $T_{a_1 \dots a_{\nu+1}}^{b_1 \dots b_\nu}$ be the coefficients of the expansion at the vectors $\partial_{b_1} \dots \partial_{b_\nu} l^\alpha$. Then the object

$$T_{ba}^c = T_{a_1 \dots a_{\nu-1} ba}^{a_1 \dots a_{\nu-1} c} \quad (18)$$

is transformed according to the law

$$\begin{aligned} T_{b'a'}^{c'} &= A_c^{c'} A_{b'}^b A_a^a T_{ba}^c + m^{(\nu-1)} A_e^{c'} \partial_{a'} A_{b'}^e \\ &+ 2(\nu-1) m^{(\nu-2)} A_e^e \delta_{(a'}^{c'} \partial_{b')} A_{e'}^e + P(\nu-2) \delta_{x'}^{c'} A_e^{e'} \partial_{b'} A_{e'}^e, \end{aligned} \quad (19)$$

where

$$P(\nu) = \sum_{k=0}^{\nu-1} (\nu-k) m^{\nu-k-1}. \quad (20)$$

It follows from this that the object

$$G_{ba}^c = \frac{1}{m^{(\nu-1)}} T_{ba}^c - \frac{2(\nu-1)}{mP(\nu)} T_{(a} \delta_{b)}^c - \frac{P(\nu-2)}{m^{(\nu-1)} P(\nu)} T_b \delta_a^c, \quad (21)$$

$$T_b = T_{eb}^e \quad (22)$$

is the object of the affine connection in X_m corresponding to the m -surface.

Using the connection found, we obtain the vectors

$$l_{a_1 \dots a_i}^\alpha = \nabla_{a_1} \dots \nabla_{a_i} l^\alpha \quad (i = 1, \dots, \nu). \quad (23)$$

The vectors

$$l_{a_1 \dots a_i}^\alpha; \quad n_{p_k}^\alpha \quad (i = 1, \dots, \nu; \quad k = 1, \dots, s). \quad (24)$$

determine an invariant normalization of the m -surface. The covariant vectors $L_{\alpha}^{a_1 \dots a_i}; \quad N_{\alpha}^{p_k}$ ($i = 1, \dots, \nu; \quad k = 1, \dots, s$), where the covectors $L_{\alpha}^{a_1 \dots a_i}$ are determined by the conditions

$$L_{\alpha}^{a_1 \dots a_i} l^{\alpha} = 0; \quad L_{\alpha}^{a_1 \dots a_i} n_{p_k}^{\alpha} = 0; \quad L_{\alpha}^{a_1 \dots a_i} l_{b_1 \dots b_i}^{\alpha} = \delta_{b_1 \dots b_i}^{a_1 \dots a_i}; \quad L_{\alpha}^{a_1 \dots a_i} l_{b_1 \dots b_j}^{\alpha} = 0 \quad (25)$$

and form the basis reciprocal to the basis (24).

It is not difficult to show that the derivative formulas of the surface under consideration have the form

$$\begin{aligned} \nabla_a l_{a_1 \dots a_i}^{\alpha} &= l_{aa_1 \dots a_i}^{\alpha}, \quad (i = 1, \dots, \nu - 1), \\ \nabla_a l_{a_1 \dots a_{\nu}}^{\alpha} &= \sum_{i=1}^{\nu} h_{aa_1 \dots a_{\nu}}^{b_1 \dots b_i} l_{b_1 \dots b_i}^{\alpha} + H_{aa_1 \dots a_{\nu}}^{p_1} n_{p_1}^{\alpha}, \quad (26) \end{aligned}$$

$$\nabla_a n_{p_j}^{\alpha} = \sum_{i=1}^{\nu} K_{ap_j}^{b_1 \dots b_i} l_{b_1 \dots b_i}^{\alpha} + \sum_{k=1}^{j+1} M_{ap_j}^{q_k} n_{q_k}^{\alpha}$$

(where the summation over k for $j = s$ is carried out up to $k = s$).

Thus we obtain:

Theorem. An m -dimensional surface of differential class $(v + s)$ is determined in an affine space, up to an automorphism, by specifying the objects of the system of differential equations (26) that satisfy the integrability conditions.

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