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P. P. Zabreiko

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Abstract

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MATHEMATICS

P. P. ZABREIKO

ON THE DIFFERENTIABILITY OF NONLINEAR OPERATORS IN THE SPACES $\mathcal{L}_p$

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Many methods for the study of nonlinear operator equations (the theory of implicit functions, approximate and topological methods, etc.) require smoothness of the nonlinear operators entering these equations. In connection with this, the problem arises of differentiating specific operators, for example integral operators, acting from one space E_1 into another space E_2 . In the present note the question of differentiability is solved for the Uryson operator

$$Ax(t) = \int_{\Omega_1} K[t, s, x(s)] ds, \quad (1)$$

acting from the space $\mathcal{L}_p = \mathcal{L}_p(\Omega_1)$ ($1 \leq p \leq \infty$) into the space $\mathcal{L}_q = \mathcal{L}_q(\Omega_2)$ ($0 < q \leq \infty$); here, as usual, Ω_1 and Ω_2 are sets of finite Lebesgue measure. Some results on the differentiability of the operator (1) were obtained earlier in (¹⁻⁵).

Below we use certain special spaces of functions of two variables. Denote by N_p ($1 \leq p \leq \infty$) the Banach space of functions $u(t, s)$ ($t \in \Omega_2, s \in \Omega_1$) for which the norm is meaningful and finite

$$\|u(t, s)\|_{N_p} = \left\| \sup_{t \in \Omega_2} |u(t, s)| \right\|_{\mathcal{L}_p(\Omega_1)}. \quad (2)$$

Next, denote by $B_{p,q}$ ($B_{p,q}^0$) the space of functions $k(t, s)$ ($t \in \Omega_2, s \in \Omega_1$) for which the integral operators

$$Kx(t) = \int_{\Omega_1} k(t, s)x(s) ds, \quad |K|x(t) = \int_{\Omega_1} |k(t, s)|x(s) ds \quad (3)$$

act from $\mathcal{L}_p$ to $\mathcal{L}_q$ and are continuous (completely continuous). We introduce the norm (6) in the spaces $B_{p,q}$ and $B_{p,q}^0$ by the equality

$$\|k(t, s)\|_{B_{p,q}} = \sup_{\|x\|_{\mathcal{L}_p} \leq 1} \left\| \int_{\Omega_1} |k(t, s)x(s)| ds \right\|_{\mathcal{L}_q}. \quad (4)$$

1. Suppose that the functions $K(t, s, u)$ ($t \in \Omega_2$, $s \in \Omega_1$, $-\infty < u < \infty$) are measurable on $\Omega_2 \times \Omega_1$ for all $u \in (-\infty, \infty)$ and for almost all $\{t, s\} \in \Omega_2 \times \Omega_1$ have a derivative with respect to u that is continuous. Introduce the superposition operator

$$\mathfrak{G}u(t, s) = K'_u[t, s, u(t, s)]. \quad (5)$$

Theorem 1. Let $A\theta \in \mathcal{L}_q$ and let the operator (5) act from N_p into $B_{p,q}$ and be continuous.

Then the nonlinear integral operator (1) acts from $\mathcal{L}_p$ into $\mathcal{L}_q$ and is continuously differentiable on $\mathcal{L}_p$, moreover

$$A'(x_0)h = \int_{\Omega_1} K'_u[t, s, x_0(s)]h(s) ds \quad (x_0, h \in \mathcal{L}_p). \quad (6)$$

The main assumption of Theorem 1 is the continuity of the superposition operator (5). This assumption, for example, is satisfied if the inequality

$$|K'_u(t, s, u_1) - K'_u(t, s, u_2)| \leq \sum_{i=1}^n R_i(t, s, u) |u_1 - u_2|^{\delta_i} \quad (|u_1|, |u_2| \leq u), \quad (7)$$

holds, where $0 < \delta_1 < \dots < \delta_n < p-1$, and the functions $R_i(t, s, u)$ ($i = 1, \dots, n$) define operators acting from N_p into $B_{\frac{p}{1+\delta_i}, q}$ and bounded operators $\mathfrak{R}_i u(t, s) = R_i[t, s, u(t, s)]$. More delicate criteria for the continuity of operator (5) are based on the following assertion.

Theorem 2. Let the function $Q(t, s, u)$ ($t \in \Omega_2$, $s \in \Omega_1$, $-\infty < u < \infty$) be measurable on $\Omega_2 \times \Omega_1$ for all $u \in (-\infty, \infty)$ and continuous in u for almost all $\{t, s\} \in \Omega_2 \times \Omega_1$. Let the superposition operator $\mathfrak{Q}u(t, s) = Q[t, s, u(t, s)]$ act from N_p into $B_{p,q}$, where $p > 1$, $q < \infty$. Then the operator (5) acts from N_p into $B_{p,q}^0$.

Then the superposition operator $\mathfrak{Q}$ is continuous.

From Theorems 1 and 2 it follows:

Theorem 3. Let $1 < p \leq \infty$, $0 < q < \infty$. Let $A\theta \in \mathcal{L}_q$, and let operator (5) act from N_p into $B_{p,q}^0$.

Then the assertion of Theorem 1 is valid.

Applying various criteria for complete continuity of linear integral operators, from Theorem 3 one can obtain more particular criteria of differentiability. We give one example.

Theorem 4. Let $1 < p \leq \infty$, $0 < q < \infty$, and let $A\theta \in \mathcal{L}_q$. Let the function $K'_u(t, s, u)$ satisfy the inequality

$$|K'_u(t, s, u)| \leq \sum_{j=1}^m S_j(t, s) g_j(s, u), \quad (8)$$

where $g_j(s, u)$ ($j = 1, \dots, m$) are nonnegative functions satisfying the Carathéodory conditions, and each operator $g_j = g_j[s, x(s)]$ acts from $\mathcal{L}_p$ into some $\mathcal{L}_{r_i}$, $1/r_i \leq 1 - 1/p$, while $S_j(t, s) \in B_{\frac{r_i p}{r_i + p}, q}$ for $r_i < \infty$, or $S_j(t, s) \in B_{p, q}^0$ for $r_i = \infty$.

Then the assertion of Theorem 1 is valid.

2. We give a theorem on the differentiability of operator (1) at a fixed point. Suppose that the function $K(t, s, u)$ ($t \in \Omega_2$, $s \in \Omega_1$, $-\infty < u < \infty$) is measurable on $\Omega_2 \times \Omega_1$ for all $u \in (-\infty, \infty)$ and is continuous in u for almost all $\{t, s\} \in \Omega_2 \times \Omega_1$. Let $x_0(s)$ be a fixed function, and suppose that there exists

$$\lim_{u \rightarrow 0} \frac{1}{u} \{K[t, s, x_0(s) + u] - K[t, s, x_0(s)]\} = K_0(t, s), \quad (9)$$

Introduce the superposition operator

$$\mathfrak{G}_0 u(t, s) = G_0[t, s, u(t, s)], \quad (10)$$

$$G_0(t, s, 0) = K_0(t, s), \quad G_0(t, s, u) = \frac{1}{u} \{K[t, s, x_0(s) + u] - K[t, s, x_0(s)]\}.$$

Theorem 5. Let $x_0 \in \mathcal{L}_p$, $Ax_0 \in \mathcal{L}_q$, and let operator (10) act from $\mathcal{L}_p$ into $B_{p, q}$ and be continuous at the point θ .

Then the nonlinear integral operator (1) acts from $\mathcal{L}_p$ into $\mathcal{L}_q$, is differentiable at the point x_0 , and

$$A'(x_0)h = \int_{\Omega_1} K_0(t, s)h(s) ds. \quad (11)$$

Operator (10) acts from $\mathcal{L}_p$ into $B_{p, q}$ and is continuous if $p > 1$, $q < \infty$ and if it acts from N_p into $B_{p, q}$. In particular, the latter is ful-

when the inequality holds

$$|G_0(t, s, u) - K_0(t, s)| \leq \sum_{j=1}^m S_j(t, s) |u|^{\delta_j}, \quad (12)$$

where $0 = \delta_0 < \delta_1 < \dots < \delta_m \leq p - 1$, $S_0(t, s) \in B_{p,q}^0$ and $S_j(t, s) \in B_{\frac{p}{1+\delta_j},q}$ ($j = 1, \dots, m$).

3. We give one theorem on the differentiability of the operator (1) on dense sets. Suppose, as in Sec. 1, that the function $K(t, s, u)$, for almost all $\{t, s\} \in \Omega_2 \times \Omega_1$, has a derivative $K'_u(t, s, u)$ continuous in u .

Theorem 6. *Let $1 < p < p_0 \leq \infty$, $0 < q < \infty$. Let $A\theta \in \mathcal{L}_q$ and let the superposition operator*

$$\mathfrak{H}(u, v) = H[t, s, u(t, s), v(t, s)], \quad (13)$$

where $H(t, s, u, v) = \frac{1}{1+|v|} \{K(t, s, u+v) - K(t, s, u)\}$, act from $N_{p_0} \times N_p$ into $B_{p,q}^0$. Suppose the operator (5) acts from N_{p_0} into $B_{p,q}^0$.

Then the nonlinear integral operator (1) acts from $\mathcal{L}_p$ into $\mathcal{L}_q$, is continuous, differentiable at every point of $\mathcal{L}_{p_0}$, and

$$A'(x_0)h = \int_{\Omega_1} K'_u[t, s, x_0(s)] h(s) ds \quad (x_0 \in \mathcal{L}_{p_0}, h \in \mathcal{L}_p). \quad (14)$$

Suppose, for example, that $K(t, s, u)$ and $K'_u(t, s, u)$ satisfy the inequalities

$$|K(t, s, u) - K(t, s, 0)| \leq \sum_{i=1}^n R_i(t, s) |u|^{k_i}, \quad (15)$$

$$|K'_u(t, s, u)| \leq \sum_{j=1}^m S_j(t, s) |u|^{l_j}. \quad (16)$$

Then the conditions of Theorem 6 are satisfied if:

- 1°. $1 < p < p_0 \leq \infty$, $0 < q < \infty$, $A\theta \in \mathcal{L}_q$.
- 2°. For each $i = 1, \dots, n$, one of the following relations holds:
 - a) $p_0 < \infty$, $k_i = 0$, and $R_i(t, s) \in B_{p,q}^0$;
 - b) $p_0 < \infty$, $0 < k_i < \frac{p_0}{p_0 - p}$, and

$$R_i(t, s) \in B_{\frac{pp_0}{pk_i+p_0}, q};$$

$$\text{c) } p_0 < \infty, \frac{p_0}{p_0 - p} \leq k_i, \text{ and } R_i(t, s) \in B_{\frac{p}{k_i}, q};$$

$$\text{d) } p_0 = \infty, 0 \leq k_i \leq 1, \text{ and } R_i(t, s) \in B_{p, q}^0;$$

$$\text{e) } p_0 = \infty, 1 < k_i, \text{ and } R_i(t, s) \in B_{\frac{p}{k_i}, q}.$$

3°. For each $j = 1, \dots, m$, one of the following relations holds:

$$\text{a) } p_0 < \infty, l_j = 0, \text{ and } S_j(t, s) \in B_{p, q}^0;$$

$$\text{b) } p_0 < \infty, l_j > 0, \text{ and } S_j(t, s) \in B_{\frac{pp_0}{pl_j+p_0}, q};$$

$$\text{c) } p_0 = \infty, \text{ and } S_j(t, s) \in B_{p, q}^0.$$

4. For the nonlinear integral operator

$$Ax(t) = \int_{\Omega_1} K(t, s) f[s, x(s)] ds \quad (17)$$

the assertions of Secs. 1-3 can be strengthened, since it can be represented in the form of a superposition of the nonlinear operator $fx = f[s, x(s)]$ and a linear integral operator. The differentiability conditions for the superposition operator f have been studied in detail in (2). We give here two further assertions.

Theorem 7. *Let $1 \leq r < p \leq \infty$, $x_0 \in \mathcal{L}_p$, $fx_0 \in \mathcal{L}_r$, and suppose there exists a function*

$$g(s) = \lim_{u \rightarrow 0} \frac{1}{u} \{f[s, x_0(s) + u] - f[s, x_0(s)]\}, \quad g(s) \in \mathcal{L}_{\frac{pr}{p-r}}.$$

Suppose

$$|f[s, x_0(s) + u] - f[s, x_0(s)] - g(s)u| \leq c(s)|u|^k \quad (|u| \leq u_0),$$

where $1 \leq k < \infty$, $c(s) \in \mathcal{L}_\mu$ and

$$|f[s, x_0(s) + u] - f[s, x_0(s)]| \leq \sum_{i=1}^n c_i(s)|u|^{\gamma_i} + b|u|^{p/r} \quad (-\infty < u < \infty),$$

where $0 \leq \gamma_0 < \gamma_1 < \dots < \gamma_n \leq 1$, $c_i(s) \in \mathcal{L}_{\mu_i}$. Suppose that for each $i = 1, \dots, n$

$$\frac{1}{r} - \frac{1}{p} \geq \frac{1}{\mu_i}, \quad \frac{1}{r} - \frac{1}{p} \geq \frac{k-1}{k-\gamma_i} \frac{1}{\mu_i} + \frac{1-\gamma_i}{k-\gamma_i} \frac{1}{\mu}.$$

Then the operator $fx = f[s, x(s)]$ maps $\mathcal{L}_p$ into $\mathcal{L}_r$, is differentiable at the point x_0 , and $f'(x_0)h = g(s)h(s)$ ($h \in \mathcal{L}_p$).

Theorem 8. Let $1 \leq r < p < p_0 \leq \infty$. Suppose that for almost all $s \in \Omega_1$ the function $f(s, u)$ has a derivative $f'_u(s, u)$ continuous in u , and that the superposition operator $\mathfrak{G}x = f'_u[s, x(s)]$ maps $\mathcal{L}_{p_0}$ into $\mathcal{L}_{\frac{pr}{p-r}}$.

Suppose that $f\theta \in \mathcal{L}_q$ and that one of the following conditions is satisfied:

a) $p_0 < \infty$ and

$$\frac{|f(s, u+v) - f(s, u)|}{1+|v|} \leq a(s) + b_1|u|^{p_0 \frac{p-r}{pr}} + b_2|v|^{\frac{p-r}{r}},$$

where $a(s) \in \mathcal{L}_{\frac{pr}{p-r}}$, and b_1, b_2 are constants;

b) $p_0 = \infty$ and for every $R > 0$

$$\frac{|f(s, u+v) - f(s, u)|}{1+|v|} \leq a_R(s) + b_R|v|^{\frac{p-r}{r}} \quad (|u| \leq R),$$

where $a_R(s) \in \mathcal{L}_{\frac{pr}{p-r}}$ and b_R is a constant.

Then the operator $fx = f[s, x(s)]$ maps $\mathcal{L}_p$ into $\mathcal{L}_r$, is continuous, and is differentiable at every point of the space $\mathcal{L}_{p_0}$:

$$f'(x_0)h = \mathfrak{G}x_0(s) \cdot h(s)$$

($x_0 \in \mathcal{L}_{p_0}$, $h \in \mathcal{L}_p$).

Suppose, for example, that the function $f(s, u)$ satisfies the inequality

$$|f(s, u) - f(s, 0)| \leq \sum_{i=1}^n c_i(s) |u|^{k_i}, \quad (18)$$

where $0 \leq k_0 < k_1 < \dots < k_n$, $c_i(s) \in \mathcal{L}_{\mu_i}$. Then the conditions of Theorem 8 are fulfilled if $1 \leq r < p < p_0 \leq \infty$, $f(s, 0) \in \mathcal{L}_r$, for each $i = 1, \dots, n$,

$$\frac{1}{\mu_i} + \frac{k_i}{p} \leq \frac{1}{r}, \quad \frac{1}{\mu_i} + \frac{k_i}{p_0} \leq \frac{1}{r} - \frac{1}{p}$$

and if the operator $\mathfrak{G}x = f'_u[s, x(s)]$ maps $\mathcal{L}_{p_0}$ into $\mathcal{L}_{\frac{pr}{p-r}}$.

5. Theorems of §§ 1–4 are easily carried over to nonlinear integral operators acting in spaces of vector-functions.

Above, derivatives of nonlinear operators in the Fréchet sense were studied. Assertions analogous to Theorems 1–8 can also be proved for Gateaux derivatives.

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Voronezh State
University

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Note: Figure translations are in progress. See original paper for figures.

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