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Abstract

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MATHEMATICS

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ASYMPTOTIC BEHAVIOR OF THE SPECTRAL MATRIX OF A ONE-DIMENSIONAL DIRAC SYSTEM

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1. In relativistic Dirac quantum theory the wave function Ψ satisfies an equation of the form

$$(p_0 + Vc^{-1} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta)\Psi = 0, \quad (1)$$

where $V = V(x, y, z)$ is the potential function, c is the speed of light,

$$p_0 = i\hbar c^{-1} \frac{\partial}{\partial t}, \quad p_x = -i\hbar \frac{\partial}{\partial x}, \quad p_y = -i\hbar \frac{\partial}{\partial y}, \quad p_z = -i\hbar \frac{\partial}{\partial z},$$

$\hbar$ is Planck's constant, and the operators $\alpha_x, \alpha_y, \alpha_z$ and β satisfy the conditions

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1, \quad \alpha_x \alpha_y + \alpha_y \alpha_x = \alpha_x \alpha_z + \alpha_z \alpha_x = \alpha_y \alpha_z + \alpha_z \alpha_y = 0,$$

$$\beta^2 = m^2 c^2, \quad \alpha_x \beta + \beta \alpha_x = \alpha_y \beta + \beta \alpha_y = \alpha_z \beta + \beta \alpha_z = 0, \quad (2)$$

and, finally, m is the mass of the particle. These formulas follow from formula (19), § 53.1 of [1], if in the latter formula we set $A = 0$ and $e\Phi = V$. Let the operators $\alpha_x, \alpha_y, \alpha_z$, and β be square 4×4 matrices. Then the wave function Ψ will be a 4×1 matrix, i.e., a four-component function

$$\Psi = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}, \quad \Psi_k = \Psi_k(x, y, z; t) \quad (k = 1, 2, 3, 4),$$

and therefore equation (1) is equivalent to a system of four partial differential equations for the functions $\Psi_k = \Psi_k(x, y, z; t)$ ($k = 1, 2, 3, 4$). If we put $\Psi = e^{-iWt/\hbar} X$, where W is the energy, and the four-component function X depends only on x, y , and z , then equation (1) takes the form

$$\{(W - V)c^{-1} + \alpha_x p_x + \alpha_y p_y + \alpha_z p_z + \beta\}X = 0. \quad (3)$$

Here W plays the role of the eigenvalue parameter.

2. We shall consider the one-dimensional case of system (3), i.e., assume that the functions V and X depend only on x . Then from system (3) we obtain

$$\{(W - V)c^{-1} + \alpha p_x + \beta\}X = 0, \quad (4)$$

where

$$p_x = -i\hbar \frac{d}{dx}, \quad \alpha^2 = 1, \quad \beta^2 = m^2 c^2, \quad \alpha\beta + \beta\alpha = 0. \quad (5)$$

In this case α and β are 2×2 matrices, while the function X is now two-component and may be regarded as a 2×1 matrix. Therefore equation (4) is equivalent to two simultaneous ordinary differential equations of first order for the components $X_1(x)$ and $X_2(x)$ of the function $X(x)$.

Let

$$\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \frac{\beta}{mc} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X(x) = \begin{pmatrix} X_1(x) \\ X_2(x) \end{pmatrix}.$$

Then conditions (5) are satisfied, and therefore from equation (4) it follows that

$$\begin{aligned} \{W - V(x)\}c^{-1}X_1 - \hbar X_2' + mcX_1 &= 0, \\ \{W - V(x)\}c^{-1}X_2 + \hbar X_1' - mcX_2 &= 0. \end{aligned} \quad (6)$$

Suppose that the units of measurement are chosen so that the speed of light c and Planck's constant $\hbar$ are equal to 1. Then, also putting

$$p(x) = V(x) - m, \quad r(x) = V(x) + m, \quad \lambda = W,$$

we can write system (6) in the form

$$\begin{aligned} dX_2/dx + p(x)X_1 &= \lambda X_1, \\ -dX_1/dx + r(x)X_2 &= \lambda X_2. \end{aligned} \quad (7)$$

Let us adjoin to system (7) the boundary condition

$$X_2(0) - hX_1(0) = 0, \quad (8)$$

where h is an arbitrary real number.

Denote by $\varphi(x, \lambda) = \{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ the solution of system (7) satisfying the initial conditions

$$\varphi_1(0, \lambda) = 1, \quad \varphi_2(0, \lambda) = h. \quad (8')$$

It is obvious that the vector-function $\varphi(x, \lambda) = \{\varphi_1(x, \lambda), \varphi_2(x, \lambda)\}$ satisfies the boundary condition (8).

It is known ⁽²⁻³⁾ that for a given h , to each boundary-value problem (7)–(8) there corresponds a unique* nondecreasing, bounded on every finite interval, left-continuous function $\rho(\lambda)$ ($-\infty < \lambda < \infty$), which generates an isometric mapping of the space of vector-functions $f(x) = \{f_1(x), f_2(x)\} \subset \mathcal{L}^2(0, \infty)$ square-integrable on the half-line $(0, \infty)$:

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\} dx < +\infty,$$

onto the space $\mathcal{L}_{\{\rho(\lambda)\}}(-\infty, \infty)$ by the formulas

$$F(\lambda) = \int_0^\infty \{f_1(x)\varphi_1(x, \lambda) + f_2(x)\varphi_2(x, \lambda)\} dx, \quad (9)$$

$$f_1(x) = \int_{-\infty}^\infty F(\lambda)\varphi_1(x, \lambda) d\rho(\lambda), \quad (10)$$

$$f_2(x) = \int_{-\infty}^\infty F(\lambda)\varphi_2(x, \lambda) d\rho(\lambda), \quad (11)$$

where the integrals (9) and (10)–(11) converge in the metrics of the spaces $\mathcal{L}_{\{\rho(\lambda)\}}^2(-\infty, \infty)$ and $\mathcal{L}^2(0, \infty)$, respectively, and Parseval's equality holds:

$$\int_0^\infty \{f_1^2(x) + f_2^2(x)\} dx = \int_{-\infty}^\infty F^2(\lambda) d\rho(\lambda).$$

3. Let us introduce the following terminology: the function $\rho(\lambda)$ will be called the **spectral function** of problem (7)–(8); the matrix of second order $\Theta(x, s; \lambda) = \{\Theta_{ik}(x, s; \lambda)\}$ ($i, k = 1, 2$), where the functions $\Theta_{ik}(x, s; \lambda)$

* The uniqueness of the function $\rho(\lambda)$ was proved by B. M. Levitan.

are defined by the formulas

$$\Theta_{ik}(x, s; \lambda) = \begin{cases} \int_0^\lambda \varphi_i(x, \lambda) \varphi_k(s, \lambda) d\rho(\lambda), & \lambda > 0, \\ -\int_\lambda^0 \varphi_i(x, \lambda) \varphi_k(s, \lambda) d\rho(\lambda), & \lambda < 0, \\ 0, & \lambda = 0, \end{cases} \quad (12)$$

we shall call the **spectral matrix** of problem (7)–(8).

In the present note, by means of a method analogous to B. M. Levitan's method (⁴), the asymptotic behavior of the spectral matrix $\Theta(x, s; \lambda)$ as $|\lambda| \rightarrow \infty$ is studied. We shall compare the spectral matrix $\Theta(x, s; \lambda)$ with the spectral matrix $\Theta^*(x, s; \lambda)$ of a simpler problem, namely, problem (7)–(8) with $p(x) = r(x) \equiv 0$ and $h = 0$. In this case $\varphi_1(x, \lambda) = \cos \lambda x$, $\varphi_2(x, \lambda) = \sin \lambda x$, and therefore the matrix $\Theta^*(x, s; \lambda)$ is defined by the formulas

$$\Theta_{11}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \cos \lambda x \cos \lambda s d\lambda = \frac{1}{2\pi} \left\{ \frac{\sin \lambda(x-s)}{x-s} + \frac{\sin \lambda(x+s)}{x+s} \right\},$$

$$\begin{aligned} \Theta_{12}^*(x, s; \lambda) &= \frac{1}{\pi} \int_0^\lambda \cos \lambda x \sin \lambda s d\lambda = \\ &= -\frac{1}{2\pi} \left\{ \frac{\cos \lambda(x-s)}{x-s} + \frac{\cos \lambda(x+s)}{x+s} \right\} + \frac{1}{\pi} \frac{x}{x^2 - s^2}, \end{aligned}$$

$$\begin{aligned} \Theta_{21}^*(x, s; \lambda) &= \frac{1}{\pi} \int_0^\lambda \sin \lambda x \cos \lambda s d\lambda = \\ &= \frac{1}{2\pi} \left\{ \frac{\cos \lambda(x-s)}{x-s} - \frac{\cos \lambda(x+s)}{x+s} \right\} - \frac{1}{\pi} \frac{x}{x^2 - s^2}, \end{aligned}$$

$$\Theta_{22}^*(x, s; \lambda) = \frac{1}{\pi} \int_0^\lambda \sin \lambda x \sin \lambda s d\lambda = \frac{1}{2\pi} \left\{ \frac{\sin \lambda(x-s)}{x-s} - \frac{\sin \lambda(x+s)}{x+s} \right\}.$$

Lemma. If the coefficients $p(x)$ and $r(x)$ are summable on every finite interval and (x_0, x_1) is an arbitrary finite interval, then there exists a constant $C = C(x_0, x_1)$ such that, for all values x and s belonging to the interval (x_0, x_1) , and for all a ,

$$\bigvee_a^{a+1} \{\Theta_{ik}(x, s; \lambda)\} < C \quad (i, k = 1, 2). \quad (13)$$

Estimate (13) for the matrix $\Theta^*(x, s; \lambda)$ is obtained directly, if one takes into account the explicit form of the functions $\Theta_{ik}^*(x, s; \lambda)$ ($i, k = 1, 2$).

Estimate (13) makes it possible to apply Tauber's theorem for Fourier integrals of V. A. Marchenko⁽⁵⁾. As a result, the following theorem can be proved:

Theorem 1. If the coefficients $p(x)$ and $r(x)$ satisfy Dini's condition, then for every fixed x and s the following asymptotic formulas hold

$$\lim_{\lambda \rightarrow \infty} \left\{ \Theta_{11}(x, s; \lambda) - \Theta_{11}(x, s; -\lambda) - \frac{1}{\pi} \left[\frac{\sin \lambda(x-s)}{x-s} + \frac{\sin \lambda(x+s)}{x+s} \right] + \frac{4}{\pi} \frac{\sin \lambda(x-s)}{x-s} \sin^2 \left[\frac{1}{4} \int_s^x \{p(\tau) + r(\tau)\} d\tau \right] \right\} = 0, \quad (14)$$

$$\lim_{\lambda \rightarrow \infty} \left\{ \Theta_{12}(x, s; \lambda) - \Theta_{12}(x, s; -\lambda) - \frac{2}{\pi} \frac{\sin \lambda(x-s)}{x-s} \sin \left[\frac{1}{2} \int_s^x \{p(\tau) + r(\tau)\} d\tau \right] \right\} = 0, \quad (15)$$

$$\lim_{\lambda \rightarrow \infty} \left\{ \Theta_{21}(x, s; \lambda) - \Theta_{21}(x, s; -\lambda) + \frac{2}{\pi} \frac{\sin \lambda(x-s)}{x-s} \sin \left[\frac{1}{2} \int_s^x \{p(\tau) + r(\tau)\} d\tau \right] \right\} = 0, \quad (16)$$

$$\lim_{\lambda \rightarrow \infty} \left\{ \Theta_{22}(x, s; \lambda) - \Theta_{22}(x, s; -\lambda) - \frac{1}{\pi} \left[\frac{\sin \lambda(x-s)}{x-s} - \frac{\sin \lambda(x+s)}{x+s} \right] + \frac{4}{\pi} \frac{\sin \lambda(x-s)}{x-s} \sin^2 \left[\frac{1}{4} \int_s^x \{p(\tau) + r(\tau)\} d\tau \right] \right\} = 0, \quad (17)$$

The asymptotic formulas (14)–(17) hold uniformly in each finite domain of variation of the variables x and s .

From Theorem 1, more precisely from formula (14), for $x = s = 0$ it follows (by virtue of (8') and (12))

Theorem 2. If the coefficients $p(x)$ and $r(x)$ satisfy Dini's condition, then the asymptotic formula

$$\lim_{\lambda \rightarrow \infty} \left\{ \rho(\lambda) - \rho(-\lambda) - \frac{2}{\pi} \lambda \right\} = 0$$

holds.

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Note: Figure translations are in progress. See original paper for figures.

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