

SOME ESTIMATES FOR A SINGULAR INTEGRAL \

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Abstract

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MATHEMATICS

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SOME ESTIMATES FOR A SINGULAR INTEGRAL *

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Let the function $u(x)$ be defined and continuous on the interval (a, b) . Introduce the functions

$$\Omega(u, r_1, r_2) = \max_{x \in [a+r_1, b-r_2]} |u(x)|,$$

$$\omega(u, \tau, r_1, r_2) = \max_{\substack{|x_1 - x_2| \leq \tau \\ x_1, x_2 \in [a+r_1, b-r_2]}} |u(x_1) - u(x_2)|,$$

where $0 < r_1, r_2, r_1 + r_2 < b - a, 0 < \tau \leq b - a - r_1 - r_2$. Denote

$$v(x) = \int_a^b \frac{u(s)}{s - x} ds,$$

where the integral is understood in the sense of the Cauchy principal value.

Theorem 1. If the integrals

$$\begin{aligned} & \int_0^t \Omega(u, t, (b-a)/2) dt, & \int_0^t \Omega(u, (b-a)/2, t) dt, \\ & \int_0^t \frac{\omega(u, t, r_1/2, (b-a)/4)}{t} dt, & \int_0^t \frac{\omega(u, t, (b-a)/4, r_2/2)}{t} dt \end{aligned}$$

converge, then the estimate

$$\begin{aligned} \Omega(v, r_1, r_2) \leq C_1 & \left[\frac{1}{r_1} \int_0^{r_1/2} \Omega(u, t, (b-a)/2) dt + \frac{1}{r_2} \int_0^{r_2/2} \Omega(u, (b-a)/2, t) dt \right. \\ & + \int_0^{r_1/2} \frac{\omega(u, t, r_1/2, (b-a)/4)}{t} dt + \int_0^{r_2/2} \frac{\omega(u, t, (b-a)/4, r_2/2)}{t} dt \\ & \left. + \int_{r_1/2}^{(b-a)/2} \frac{\Omega(u, t(b-a)/4)}{t} dt + \int_{r_2/2}^{(b-a)/2} \frac{\Omega(u, (b-a)/4, t)}{t} dt \right], \end{aligned}$$

where C_1 is a constant independent of $u(x)$.

Theorem 2. If the integrals

$$\int_0^{\delta} \omega(u, r_2/2, r_1/2, t) dt, \quad \int_0^{\delta} \omega(u, r_1/2, t, r_2/2) dt, \quad \int_0^{\delta} \frac{\omega(u, t, r_1/2, r_2/2)}{t} dt,$$

converge, then the estimate

$$\begin{aligned} \omega(v, \delta, r_1, r_2) \leq C_2 & \left[\delta \int_{\varepsilon}^{(b-a)/2} \frac{\omega(u, t, r_1/2, r_2/2)}{t^2} dt \right. \\ & + \frac{\delta}{r_1^2} \int_0^{r_1/2} \omega(u, r_1/2, t, r_2/2) dt + \frac{\delta}{r_2^2} \int_0^{r_2/2} \omega(u, r_2/2, r_1/2, t) dt + \\ & \left. + \int_0^{\delta} \frac{\omega(u, t, r_1/2, r_2/2)}{t} dt + \left(\frac{\delta}{r_1} + \frac{\delta}{r_2} \right) \Omega(u, r_1, r_2) \right], \end{aligned}$$

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$$+ \int_0^{\delta} \frac{\omega(u, t, r_1/2, r_2/2)}{t} dt + \left(\frac{\delta}{r_1} + \frac{\delta}{r_2} \right) \Omega(u, r_1, r_2) \Big],$$

where $\varepsilon = \min\{\delta, r_1/2, r_2/2\}$, and C_2 is a constant independent of $u(x)$.

Let us consider some constructions based on the preceding estimates.

Suppose: a) $\varphi(\delta)$ is continuous, positive, and monotonically increasing for $0 < \delta \leq b-a$, $\lim_{\delta \rightarrow 0} \varphi(\delta) = 0$, and $\lim_{\delta \rightarrow 0} \varphi(\delta)/\delta > 0$; b) the positive continuous functions $\varphi_1(r_1)$, $\tilde{\varphi}_1(r_1)$, $\varphi_2(r_2)$, $\tilde{\varphi}_2(r_2)$, $0 < r_1, r_2 < b-a$, increase monotonically and have limit zero at zero.

Definition. We shall say that $u(x)$, defined in (a, b) , belongs to the class $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ if there exist constants $A_1, A_2 > 0$ such that, for all possible r_1, r_2, δ ,

$$\Omega(u, r_1, r_2) \leq A_1 \frac{\tilde{\varphi}_1(r_1)}{r_1} \frac{\tilde{\varphi}_2(r_2)}{r_2}, \quad \omega(u, \delta, r_1, r_2) \leq A_2 \varphi(\delta) \frac{\varphi_1(r_1)}{r_1} \frac{\varphi_2(r_2)}{r_2}.$$

It is obvious that $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ is a linear system over the field of real numbers and, moreover, it is infinite-dimensional, since, by virtue of the condition $\lim_{\delta \rightarrow 0} \varphi(\delta)/\delta > 0$, it contains the class of functions satisfying the Lipschitz condition on $[a, b]$.

With the introduction in $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ of the norm

$$\|u\|_{\varphi, \psi_1, \psi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2} = \sup_{r_1, r_2} \frac{\Omega(u, r_1, r_2) r_1 r_2}{\tilde{\varphi}_1(r_1) \tilde{\varphi}_2(r_2)} + \sup_{r_1, r_2, \delta} \frac{\omega(u, \delta, r_1, r_2) r_1 r_2}{\varphi(\delta) \varphi_1(r_1) \varphi_2(r_2)}$$

it becomes a Banach space.

The following theorem partially solves the problem of classifying $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$.

Theorem 3. Let $\varphi(\delta)$, $\psi(\delta)$ satisfy condition a), and let $\varphi_1(r_1)$, $\tilde{\varphi}_1(r_1)$, $\varphi_2(r_2)$, $\tilde{\varphi}_2(r_2)$ and $\psi_1(r_1)$, $\tilde{\psi}_1(r_1)$, $\psi_2(r_2)$, $\tilde{\psi}_2(r_2)$ satisfy condition b).

1) If

$$\varphi(\delta) \sim \psi(\delta)^*, \quad \varphi_1(r_1) \sim \psi_1(r_1), \quad \tilde{\varphi}_1(r_1) \asymp \tilde{\psi}_1(r_1),$$

$$\varphi_2(r_2) \sim \psi_2(r_2), \quad \tilde{\varphi}_2(r_2) \sim \tilde{\psi}_2(r_2),$$

then $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ and $H_{\psi, \psi_1, \psi_2}^{\tilde{\psi}_1, \tilde{\psi}_2}$ coincide.

2) If

$$\lim_{\delta \rightarrow 0} \frac{\varphi(\delta)}{\psi(\delta)} = 0, \quad \lim_{r_1 \rightarrow 0} \frac{\varphi_1(r_1)}{\psi_1(r_1)} = 0, \quad \lim_{r_2 \rightarrow 0} \frac{\varphi_2(r_2)}{\psi_2(r_2)} = 0,$$

$$\lim_{r_1 \rightarrow 0} \frac{\tilde{\varphi}_1(r_1)}{\tilde{\psi}_1(r_1)} = 0, \quad \lim_{r_2 \rightarrow 0} \frac{\tilde{\varphi}_2(r_2)}{\tilde{\psi}_2(r_2)} = 0,$$

then $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ is a proper part of $H_{\psi, \psi_1, \psi_2}^{\tilde{\psi}_1, \tilde{\psi}_2}$.

For the subsequent arguments the following will be useful.

Definition. Let a positive, continuous function $f(t)$ increase monotonically on $(0, a]$ and let $\lim_{t \rightarrow 0} f(t) = 0$. If, for some $c_1 > 1$,

$$\lim_{t \rightarrow 0} f(c_1 t)/f(t) > 1,$$

then one says that the function $f(t)$ satisfies condition (L) , and if, for some $c_2 > 1$,

$$\lim_{t \rightarrow 0} f(c_2 t)/f(t) < c_2,$$

then one says that $f(t)$ satisfies condition (L_1) .

* Positive functions $f(x)$ and $g(x)$, defined on X , are called equivalent ($f(x) \sim g(x)$) if, for some $B_1, B_2 > 0$ and every $x \in X$, $B_1 f(x) \leq g(x) \leq B_2 f(x)$.

With the help of Theorems 1 and 2 one proves

Theorem 4. *Let the functions $\varphi(\delta)$, $\tilde{\varphi}_1(r_1)$, $\tilde{\varphi}_2(r_2)$ satisfy conditions (L) and (L_1) , and let the functions $\varphi_1(r_1)$, $\varphi_2(r_2)$ satisfy condition (L) . Suppose, furthermore, that*

$$\varphi(t)\varphi_1(t) \sim \tilde{\varphi}_1(t), \quad \varphi(t)\varphi_2(t) \sim \tilde{\varphi}_2(t).$$

Then the operator

$$Au = \int_a^b \frac{u(s)}{s-x} ds$$

acts in $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$ and is bounded.

The class $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$, corresponding to the functions $\varphi(\delta) = \delta^\gamma$, $\tilde{\varphi}_1(r_1) = r_1^{1-\alpha}$, $\tilde{\varphi}_2(r_2) = r_2^{1-\beta}$, $\varphi_1(r_1) = r_1^{1-\alpha-\gamma}$, $\varphi_2(r_2) = r_2^{1-\beta-\gamma}$ ($0 < \alpha, \beta, \gamma$, $\alpha + \gamma$, $\beta + \gamma < 1$), will be denoted by $H_{\alpha, \beta, \gamma}$. The invariance of $H_{\alpha, \beta, \gamma}$ with respect to the operator A was proved by another method in (2). Arguing exactly as in (3), with the help of Theorems 3 and 4, one can show that there exists an infinite set of various classes $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$, different from the classes $H_{\alpha, \beta, \gamma}$ and invariant with respect to the operator A .

With the help of Theorem 4 one can study the question of smoothness of solutions of linear and nonlinear singular integral equations. We give one theorem of this type for nonlinear singular integral equations, which is proved by the method set forth in papers (4, 5).

Theorem 5. *Let the functions $\varphi(\delta)$, $\varphi_1(r_1)$, $\varphi_2(r_2)$, $\tilde{\varphi}_1(r_1)$, $\tilde{\varphi}_2(r_2)$ satisfy the conditions of Theorem 4, and let the function $f(s, u)$, for arbitrary $s_1, s_2 \in$*

$[a + r_1, b - r_2]$, $0 < r_1, r_2$, $r_1 + r_2 < b - a$ and $u_1, u_2 \in (-\infty, +\infty)$, satisfy the condition

$$|f(s_1, u_1) - f(s_2, u_2)| \leq D \left[\varphi(|s_1 - s_2|) \frac{\varphi_1(r_1)}{r_1} \frac{\varphi_2(r_2)}{r_2} + |u_1 - u_2| \right],$$

where D is a constant. Suppose, furthermore, that $f(s, 0) \in H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$.

Then, for small λ , the equation

$$u(x) = \lambda \int_a^b \frac{f[s, u(s)]}{s - x} ds$$

has a unique solution in $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$, and this solution can be found by the method of successive approximations, starting from any element of $H_{\varphi, \varphi_1, \varphi_2}^{\tilde{\varphi}_1, \tilde{\varphi}_2}$. The successive approximations converge uniformly on every closed subinterval interior to (a, b) .

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