

SOME LIMIT THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A RANDOM RIGHT-HAND SIDE

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Abstract

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MATHEMATICS

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SOME LIMIT THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A RANDOM RIGHT-HAND SIDE

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In many physical and, in particular, radio-engineering problems it is of interest to study random processes that are solutions of differential equations whose right-hand side contains randomness. In this case it is often possible to single out in the right-hand side of the equation a small parameter ε , which may characterize the smallness of the random perturbation, its "correlation time," etc. (see ⁽¹⁾). In the present note we study the asymptotic behavior of such random processes as $\varepsilon \rightarrow 0$.

1. Let $F(x, t, \omega, \varepsilon)$ be a function with values in l -dimensional Euclidean space E^l , defined for $x \in E^l$, $t \geq 0$, $\omega \in \Omega$, $\varepsilon \geq 0$; here Ω is the space of elementary events, on the σ -algebra $\mathfrak{A}$ of measurable subsets of which a probability measure P is given. Suppose that $F(x, t, \omega, \varepsilon)$, for fixed x, ε , is a random process measurable in t, ω , satisfies the Lipschitz condition

$$|F(x_2, t, \omega, \varepsilon) - F(x_1, t, \omega, \varepsilon)| < L|x_2 - x_1| \quad (1)$$

and, for all $t > 0$, the condition

$$\mathbf{P} \left\{ \int_0^t |F(0, s, \omega)| ds < \infty \right\} = 1. \quad (2)$$

When these requirements are fulfilled, the problem

$$dx/dt = \varepsilon F(x, t, \omega, \varepsilon); \quad x(0) = x_0, \quad (3)$$

has a unique solution, a random process $x_\varepsilon(t, \omega)$ that is continuous with probability 1.

The asymptotic behavior, as $\varepsilon \rightarrow 0$, of the solution $x_\varepsilon(t, \omega)$ of problem (3) has been considered, for example, in ⁽¹⁻³⁾.

In ⁽³⁾ it is proved that the function $x_\varepsilon(t, \omega)$ on a time interval of order $O(1/\varepsilon)$ can be uniformly approximated by the solution of the problem

$$\frac{dx}{dt} = \varepsilon \Phi^{(0)}(x); \quad x(0) = x_0 \quad \left(\Phi^{(0)}(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T M F^{(0)}(x, t, \omega) dt \right),$$

if, uniformly in x, t, ω as $\varepsilon \rightarrow 0$, the relation $F(x, t, \omega, \varepsilon) = F^{(0)}(x, t, \omega) + o(1)$ holds, and the process $F^{(0)}(x, t, \omega)$ satisfies the law of large numbers.

It follows from this result that, when $\Phi^{(0)}(x) \equiv 0$, the random process $x_\varepsilon(\varepsilon t, \omega)$ converges in probability to zero as $\varepsilon \rightarrow 0$. R. L. Stratonovich in ⁽¹⁾ first drew attention to the fact that in the still “slower” time $\tau = \varepsilon^2 t$ there is convergence to a Markov process. However, some points in the proof of this assertion given in ⁽²⁾, when F is a stationary-in-time random process, seem to us unconvincing. We shall consider this problem under other assumptions and by another method.

Let the function $F(x, t, \omega, \varepsilon)$ satisfy the conditions:

B₁. Uniformly in x, t, ω , except perhaps for a set of ω -values of probability 0, the relations

$$F(x, t, \omega, \varepsilon) = F^{(0)}(x, t, \omega) + \varepsilon F^{(1)}(x, t, \omega) + o(\varepsilon) \quad (\varepsilon \rightarrow 0),$$

$$|F^{(i)}|, |\partial F^{(i)}/\partial x|, |\partial^2 F^{(i)}/\partial x_j \partial x_k| < C \quad (i = 0, 1; j, k = 1, \dots, l).$$

B₂. There exists a family of σ -algebras N_s^t ($0 \leq s \leq t \leq \infty$) of subsets of Ω such that $N_s^t \subset \mathfrak{A}$; $N_s^t \subset N_{s_1}^{t_1}$ if $s_1 \leq s$, $t \leq t_1$, and for all $x \in E_n$, $t \geq 0$ the random variables $F^{(i)}(x, t, \omega)$ are N_t^t -measurable. (For example, if $F^{(i)}(x, t, \omega) = F^{(i)}(x, \xi(t, \omega))$, then one may take as N_s^t the σ -algebra of events generated by events of the form $\{\xi(u, \omega) \in A\}$, $s \leq u \leq t$.) Moreover, for any $t \geq 0$, $B \in N_{t+\tau}^\infty$, and for some function $\beta(\tau)$ such that, as $\tau \rightarrow \infty$, the function $\tau^6 \beta(\tau) \downarrow 0$, with probability 1,

$$|P\{B/N_0^t\} - P(B)| < \beta(\tau). \quad (4)$$

(Condition (4) was considered by Ibragimov in ^(4,5).)

B₃. Uniformly in $x, t_0 > 0$ the limits

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} M F^{(1)}(x, t, \omega) dt = \Phi^{(1)}(x),$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \int_{t_0}^{t_0+T} \text{cov}(F_j^{(0)}(x, s, \omega), F_k^{(0)}(x, t, \omega)) ds dt = a_{jk}(x), \quad (5)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} ds \int_{t_0-T}^s M \left\{ \frac{\partial F^{(0)}}{\partial x}(x, s, \omega) F^{(0)}(x, t, \omega) \right\} dt = K(x),$$

exist, and the integrals

$$\int_0^T M F^{(0)}(x, t, \omega) dt \quad \text{and} \quad \int_0^T \frac{\partial}{\partial x} M F^{(0)}(x, t, \omega) dt$$

are bounded uniformly in x, T .

B₄. There exists a sequence $T_n \rightarrow \infty$, growing no faster than a geometric progression and such that, as $n \rightarrow \infty$,

$$\delta(T_n) = \sup_{x \in E^t, t_0 > T_n} \left| T_n^6 \int_{t_0}^{t_0+T_n} M F^{(0)}(x, t, \omega) dt \right| \downarrow 0.$$

Theorem 1. *If the function $F(x, t, \omega, \varepsilon)$ satisfies conditions B_1 – B_2 , then the process $x_\varepsilon(\varepsilon^2 t)$ on the time interval $0 \leq \varepsilon^2 t \leq \tau_0$ converges weakly as $\varepsilon \rightarrow 0$ to a continuous, with probability 1, Markov process $X^{(0)}(t, \omega)$, for which*

$$M\{\Delta X^{(0)}(t, \omega) \mid X^{(0)}(t, \omega) = x\} = (K(x) + \Phi^{(1)}(x))\Delta t + o(\Delta t),$$

$$M\{\Delta X_j^{(0)}(t, \omega) \Delta X_k^{(0)}(t, \omega) \mid X^{(0)}(t, \omega) = x\} = a_{jk}(x)\Delta t + o(\Delta t).$$

From Theorem 1 one can easily derive

Theorem 2. *Let the function F satisfy conditions B_1 , B_2 , and let*

$$F^{(i)}(x, t, \omega) = F^{(i)}(x, \xi(t, \omega)),$$

where $\xi(t, \omega)$ is a random process periodic with period θ . Suppose also that the condition

$$\int_0^\theta M F^{(0)}(x, t, \omega) dt = 0 \quad (6)$$

is satisfied.

Then the conclusion of Theorem 1 is valid, where $a_{jk}(x)$, $\Phi^{(1)}(x)$, and $K(x)$ are computed by the formulas

$$\Phi^{(1)}(x) = \frac{1}{\theta} \int_0^\theta MF^{(1)}(x, t, \omega) dt;$$

$$a_{jk}(x) = \frac{1}{\theta} \int_0^\theta ds \int_{-\infty}^{+\infty} \text{cov} \left(F_j^{(0)}(x, s, \omega), F_k^{(0)}(x, t, \omega) \right) dt,$$

$$K(x) = \frac{1}{\theta} \int_0^\theta ds \left[\int_{-\infty}^\theta \text{cov} \left(\frac{\partial F^{(0)}}{\partial x}(x, s, \omega), F^{(0)}(x, s+u, \omega) \right) du + \int_0^s \frac{\partial MF^{(0)}}{\partial x}(x, s) MF^{(0)}(x, t) dt \right]. \quad (7)$$

Remark 1. For a stationary process F , formula (7) is somewhat simplified (since condition (6) becomes the condition $MF^{(0)}(x, t, \omega) \equiv 0$, and in (7) one can pass to the limit as $\theta \rightarrow 0$). For this case they were obtained by Stratonovich in ^(1, 2).

Remark 2. Relying on Theorem 1, one can show that the conclusion of Theorem 2 is also valid in the case when the process $\xi(t, \omega)$ is not periodic, but only converges to a periodic one in a sufficiently weak sense as $t \rightarrow \infty$.

Remark 3. Theorem 1 can be regarded as a generalization of the central limit theorem for random processes satisfying a mixing condition in one form or another (cf. ^(4, 5)). This becomes clear if one sets $F(x, t, \omega) \equiv F(t, \omega)$.

2. Theorems 1 and 2 can be used, in particular, for a rigorous justification, refinement, and indication of the conditions of applicability of certain conclusions of ⁽¹⁾. Let us consider, for example, the question of parametric excitation of linear systems by random forces (see ⁽¹⁾, § 19).

The equation of motion of a system whose frequency undergoes small random perturbations $\varepsilon\xi(t, \omega)$, and whose friction coefficient is γ , has the form

$$\ddot{x} + \mu^2(1 + \varepsilon\xi(t, \omega))x + \gamma\dot{x} = 0. \quad (8)$$

If the process $\xi(t, \omega)$ is stationary, ergodic, and has zero mean, with $|\xi(t, \omega)| < C$ with probability 1, and $\gamma = \varepsilon\gamma_1$ ($\gamma_1 = \text{const}$), then, making the change of variables

$$x = e^u \cos(\mu t + \theta), \quad \dot{x} = -\mu e^u \sin(\mu t + \theta) \quad (9)$$

and applying Theorem 1.1 from ⁽³⁾, we obtain that system (8) is stable on a time interval $O(1/\varepsilon)$ for all $\gamma_1 > 0$, and the approximation to the equilibrium can be described by the equation $\ddot{x} + \mu^2 x + \varepsilon\gamma_1 \dot{x} = 0$ the more accurately, the smaller ε is.

A more interesting case is obtained if $\gamma = \varepsilon^2 \gamma_1$, where $\gamma_1 = \text{const}$. Then the stability or instability of the system can be detected only on a time interval $O(1/\varepsilon^2)$. If, in addition to the requirements listed above, the process $\xi(t, \omega)$ satisfies condition B2, then Theorem 2 can be applied to the system of equations for the process $\{u_\varepsilon(t, \omega), \theta_\varepsilon(t, \omega)\}$ obtained after the substitution (9) from (8). This theorem makes it possible to establish that on the interval $0 \leq \varepsilon^2 t \leq \tau_0$, as $\varepsilon \rightarrow 0$,

$$\{u_\varepsilon(t, \omega), \theta_\varepsilon(t, \omega)\} \rightarrow \{u_0(\varepsilon^2 t, \omega), \theta_0(\varepsilon^2 t, \omega)\},$$

where $u_0(\tau, \omega)$ and $\theta_0(\tau, \omega)$ are mutually independent Markov processes of diffusion type with constant diffusion and drift coefficients. In particular, the process $u_0(\tau, \omega)$ has diffusion coefficient $\mu^2 f(2\mu)/16$ and drift coefficient $\mu^2 f(2\mu) - 4\gamma_1/8$ (here $f(\lambda)$ is the spectral density of the process $\xi(t, \omega)$). Hence it is clear that, if

$$\mu^2 f(2\mu)/4 \geq \gamma_1, \quad (10)$$

system (8) is unstable for sufficiently small ε . If the opposite condition is fulfilled, Theorem 2 permits one to assert only that no loss of stability will occur on a time interval of order $O(1/\varepsilon^2)$. Condition (10) was obtained in ⁽¹⁾ by means of nonrigorous considerations.

3. An important point in the proof of Theorem 1 is the establishment of the estimate

$$M |x_\varepsilon(\tau + h, \omega) - x_\varepsilon(\tau, \omega)|^4 < ch^2,$$

which guarantees compactness of the family of distributions associated with the process $x_\varepsilon(\tau, \omega)$ in the space of continuous functions (see (6)). For the establishment of this estimate the following is essential.

Lemma 1. Let $\{\xi_n^{(1)}, \dots, \xi_n^{(2k)}\}$ ($n = 1, 2, \dots$) be a sequence of random vectors in $E^{(2k)}$ with zero mathematical expectation, satisfying Rosenblatt's strong mixing condition with mixing coefficient $\alpha(\tau)$. Suppose also that either the conditions

$$|\xi_n^{(i)}| < C, \quad \int_0^\infty \tau^{k-1} \alpha(\tau) d\tau < \infty,$$

or, for some $m > 2$, the conditions

$$M |\xi_n^{(i)}|^{m(2k-1)} < C, \quad \int_0^\infty \tau^{k-1} [\alpha(\tau)]^{(m-2)/m} d\tau < \infty$$

are fulfilled. Then for all $N > 0$ the estimate

$$\sum_{n_i=1}^N \left| M \left(\xi_{n_1}^{(1)} \dots \xi_{n_{2k}}^{(2k)} \right) \right| < C_1 N^k. \quad (11)$$

holds.

The proof of this lemma is close in idea to the proof of similar assertions in (7). We note that in the case $\xi_n^{(1)} = \xi_n^{(2)} = \dots = \xi_n^{(2k)} = \xi_n$, from (11) there follows the estimate

$$M(\xi_1 + \dots + \xi_N)^{2k} < C_1 N^k,$$

which, as is known, cannot be improved even for independent random variables.

For estimating the conditional moments of first and second order of the increment of the process $x_\varepsilon(\varepsilon^2 t)$, the following lemma, easily following from (4), is useful.

Lemma 2. Let the family of σ -algebras N_s^t satisfy condition (4), let ξ be an N_0^t -measurable random variable, and let $G(x, \omega)$, $|G| < 1$, be a measurable function on $E^l \times \Omega$ which, for each fixed x , is an $N_{t+\tau}^\infty$ -measurable random variable. Let $g(x) = MG(x, \omega)$. Then with probability 1

$$|M\{G(\xi(\omega), \omega)/N_0^t\} - g(\xi(\omega))| < 2\beta(\tau).$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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