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Abstract

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MATHEMATICS

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ON THE J -FUNCTOR OF CELL COMPLEXES

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The J -functor was studied by Rokhlin, Milnor, and Kervaire ^(1,2) in terms of framed manifolds and the classical J -homomorphism, and also by Atiyah, Adams, and Walker in modern terms ⁽³⁻⁷⁾. It has been well studied for spheres and projective spaces. Adams, in terms of operations Ψ_Λ^k ($\Lambda = R, C$), defined the groups $J'(X)$, estimating the order of the groups $J(X)$ from below, and showed that for a pair $X \supset Y$ there is an exact sequence

$$J'(X/Y) \rightarrow J'(X) \rightarrow J'(Y) \rightarrow 0,$$

if the analogous sequence is exact for the K_R -functor. (Recall that in ⁽⁵⁾ this sequence is written for J'' , but in ⁽⁶⁾ it is proved that $J'' = J'$.) In what follows we shall speak about the complex J -functor $J_c(X) \subset \bar{J}(X)$, whose order differs from the order of $J(X)$ only by a factor of the form 2^n . Recall that it is useful to introduce the functor $\Pi^i(X, Y)$ by means of self-maps of the sphere of degree ± 1 , and to consider the J -homomorphism $J : K_c^i(X, Y) \rightarrow \Pi^i(X, Y)$, putting $J_c^i(X, Y) = JK_c^i(X, Y)$, with $J_c(X, Y) = J_c^0(X, Y)$. Generally speaking, for the J_c -functor there need not exist an exact sequence of the pair

$$\rightarrow J_c^i(X/Y) \xrightarrow{\pi^*} J_c^i(X) \xrightarrow{i^*} J_c^i(Y) \xrightarrow{\delta} J_c^{i+1}(X/Y) \rightarrow$$

and we have only that $\text{Ker } i^* \supset \text{Im } \pi^*$, etc.

Define the homology of the J -functor:

$$H_J^i(X, Y) = \text{Ker } i^* / \text{Im } \pi^*, \quad \text{Ker } i^* \subset J_c^i(X).$$

In what follows we shall be interested in the following groups: a) $H_J^0(X, Y)$, if $X/Y = S^{4n}$; b) $\pi^* J_c^0(X/Y) \subset J_c^0(X)$, $X/Y = S^{4n}$.

1. If the homology of the complex X has no torsion, $X/Y = S^{4n}$, and $H^*(X) = H^*(Y) + H^*(S^{4n})$, then from ^(5,7) one can extract that there exists an exact sequence

$$Z_2 \rightarrow J_c(S^{4n}) \xrightarrow{\pi^*} J_c(X).$$

Consider the commutative diagram

$$\begin{array}{ccccc} K_c^0(S^{4n}) & \xrightarrow{\pi^*} & K_c^0(X) & \xrightarrow{i^*} & K_c^0(Y) \\ \downarrow J & & \downarrow J & & \downarrow J \\ \Pi^0(S^{4n}) & \xrightarrow{\pi^*} & \Pi^0(X) & \xrightarrow{i^*} & \Pi^0(Y) \end{array}$$

and put $B = \text{Ker}(Ji^*) \subset K_c^0(X)$. Let $\alpha \in B$ and $\xi = \alpha + N \in K_c(X)$, where ξ is a vector stable bundle and N is a scalar. Let T_ξ denote the Thom complex of the bundle ξ over X . From the definition of the group $B = \text{Ker}(Ji^*)$ one can extract the following: the complex T_ξ has the homotopy type

$$(S^{2N} \vee E^{2N}Y) \cup_\psi D^{2N+4n},$$

where

$$\psi : S^{2N+4n-1} \rightarrow S^{2N} \vee E^{2N}Y,$$

and E is suspension. Since $N \gg n$, we have

$$\psi \in \pi_{2N+4n-1}(S^{2N}) + \pi_{2N+4n-1}(E^{2N}Y),$$

although the decomposition into a direct sum is not unique. Put $X = Y \cup_\varphi D^{4n}$, where $\varphi \in \pi_{4n-1}Y$ and $\psi = \psi_0 + \psi_1$, $\psi_0 \in \pi_{2N+4n-1}(S^{2N})$, $\psi_1 \in \pi_{2N+4n-1}(E^{2N}Y)$.

Lemma 1. $\psi_1 = E^N\varphi$, $E^N : \pi_{4n-1}(Y) \rightarrow \pi_{2N+4n-1}(E^{2N}Y)$.

We do not give the proof of Lemma 1 (see (8), § 7). Let us note that the element ψ_1 is determined uniquely in view of the canonical projection $S^{2N} \vee E^{2N}Y \rightarrow E^{2N}Y$, whereas the element ψ_0 is not unique, since the decomposition in $S^{2N} \vee E^{2N}Y$ has a large degree of arbitrariness. However, fixing the decomposition in b induces the fixing of the element ψ_0 and of the projection $\Delta_{\psi_0} : T_\xi \rightarrow S^{2N} \cup_{\psi_0} D^{2N+4n}$.

Put $S^{2N} \cup_{\psi_0} D^{2N+4n} = P_{(\xi, \psi_0)}$, $\xi - N \in B$. Let now $\eta \in K_c^0(P_{(\xi, \psi_0)})$ be such an element that $\text{ch } \eta = U + \lambda V$, where U, V are generators of the groups $H^{2N}(P_{(\xi, \psi_0)}, Z) = H^{2N+4n}(P_{(\xi, \psi_0)}, Z) = Z$ and $\lambda \in Q$, Q is the rational numbers. We denote $\lambda \bmod 1 \in Q/Z$ by $\lambda(\xi, \psi_0) = \lambda$.

The following substantial, though simple, lemma holds.

Lemma 2. *The number $\lambda(\xi, \psi_0) \in Q/Z$ depends only on the element $a = \xi - N \in K_c^0(X)$ such that $J \cdot i^*(a) = 0$. Moreover, the correspondence*

$$a \mapsto \lambda(a + N, \psi_0)$$

induces a single-valued homomorphism $B \rightarrow Q/Z$, where $B = \text{Ker}(J \cdot i)$.

Proof. It is easy to see that $\lambda(\xi, \psi_0)$ can depend only on the ambiguity in the choice of $\psi_0 \in \Pi(S^{4n}) = \pi_{2N+4n-1}(S^{2N})$, which also generates the kernel of the mapping $\pi^* : \Pi(S^{4n}) \rightarrow \Pi(X)$. Let $\psi_0^{(1)}, \psi_0^{(2)} \in \Pi(S^{4n})$ be constructed from ξ , where $\xi = a + N$, $a \in B$. We have two projections

$$\Delta_{\psi_0^{(1)}} : T_\xi \rightarrow P_{(\xi, \psi_0^{(1)})}, \quad \Delta_{\psi_0^{(2)}} : T_\xi \rightarrow P_{(\xi, \psi_0^{(2)})}.$$

In cohomology we see that the element $\bar{V} = \Delta_{\psi_0^{(1)}}^* V \in H^{4n}(X, Z)$ is determined uniquely, independently of the choice of ψ_0 . Obviously, we have

$$\text{ch}(\Delta_{\psi_0^{(1)}} \eta_1) - \text{ch}(\Delta_{\psi_0^{(2)}} \eta_2) = (\lambda_1 - \lambda_2) \bar{V},$$

where $\eta_i \in K_c^0(P_{(\xi, \psi_0^{(i)})})$, $i = 1, 2$, are elements such that $\text{ch} \eta_i = U + \lambda_i V$.

Since the element

$$\Delta_{\psi_0^{(1)}} \eta_1 - \Delta_{\psi_0^{(2)}} \eta_2 \in K_c^0(X)$$

has filtration $\geq 2n = \dim \bar{V}/2$, we see from complex Bott periodicity that $\lambda_1 - \lambda_2$ is an integer. Therefore the numbers $\lambda(\xi, \psi_0^{(1)})$ and $\lambda(\xi, \psi_0^{(2)}) \in Q/Z$ coincide. Thus the mapping $B \rightarrow Q/Z$, defined by $\lambda(\xi, \psi_0)$, is well defined. It is not difficult to show that it is additive. The lemma is proved.

From the construction of the mapping $\lambda(\xi, \psi_0) = \lambda(\xi) = \lambda(a)$, $a \in B$, $\xi = a + N$, it is clear that this mapping $\lambda : B \rightarrow Q/Z$ gives rise to a mapping $\lambda : J(B) \rightarrow Q/Z$, where $J(B) \in \Pi(X)$, and $J(B) = \text{Ker} i^* \subset J_c(X)$. Consider in the group B the subgroup $B' = \pi^* K_c^0(S^{4n})$. We have the composition $\lambda \cdot \pi^* : K_c^0(S^{4n}) \rightarrow Q/Z$.

Lemma 3. *The mapping $\lambda \cdot \pi^* : K_c^0(S^{4n}) \rightarrow Q/Z$ decomposes into the composition $\lambda \cdot \pi^* = \bar{\lambda} \cdot J$*

$$K_c^0(S^{4n}) \xrightarrow{J} \Pi^0(S^{4n}) \xrightarrow{\bar{\lambda}} Q/Z,$$

and on the group $J_c(S^{4n}) = JK_c^0(S^{4n})$ the kernel of the mapping $\bar{\lambda}$ contains no more than two elements. Moreover, the mapping $\bar{\lambda}$ decomposes into the composition $\bar{\lambda} = \bar{\lambda}' \cdot \pi^*$

$$\Pi^0(S^{4n}) \xrightarrow{\pi^*} \pi^* \Pi^0(S^{4n}) \xrightarrow{\bar{\lambda}'} Q/Z.$$

For the proof, let us note that the mapping $\lambda \pi^*$ is obviously well defined on the group $J_c(S^{4n}) = JK_c^0(S^{4n})$. The mapping $\bar{\lambda} : \Pi^0(S^{4n}) \rightarrow Q/Z$ is defined through the two-cell complex $S^{2N} \cup_h D^{2N+4n}$ by

analogy with $\lambda(\xi, \psi_0)$ for fixed ψ_0 , and, as is easy to see, the mapping $\bar{\lambda}$ is trivial on the kernel $\text{Ker } \pi^*$, since $\lambda\pi^* = \bar{\lambda}J$. The rest follows easily from comparing this with the results of Milnor, Kervaire ⁽²⁾, and Adams ^(4,5).

Remark. For two-cell complexes the mapping $\bar{\lambda}$ was, apparently, defined by Adams in connection with the question of splitting off $J(S^{4n})$ in $\Pi(S^{4n})$ as a direct summand in a work not yet published (see ⁽⁶⁾, introduction).

Theorem 1. The homomorphism

$$\bar{\lambda} : \Pi(S^{4n}) \rightarrow Q/Z$$

annihilates all compositional elements $\alpha \cdot \beta$, where $\alpha \in \Pi(S^j)$, $\beta \in \Pi(S^k)$, $j+k = 4n+1$, $j \geq 2$, $k \geq 2$, $\Pi(S^q) = \pi_{N+q-1}(S^N)$. In particular, the compositional elements in $J(S^{4n})$ form a subgroup of order at most Z_2 . For $n = 2l$, the existence in $J(S^{4n})$ of a nontrivial compositional element is equivalent to

$$|J(S^{4n})| = 2 \cdot (\text{denominator } B_k/4k),$$

where B_k is the Bernoulli number with index k .

The proof follows from the construction and properties of the homomorphism $\bar{\lambda}$ and the results of ⁽⁵⁾.

From the construction of the homomorphism $\lambda(\xi, \psi_0)$ it is clear that the “denominator” of its image in Q/Z coincides with the order of the group $J_c(S^{4n})$ up to a factor of 2. Comparing this with the definition of the group $B = \text{Ker}(Ji^*) = \text{Ker}(i^*J) \subset K_c^0(X)$ and with the fact that the homomorphism $\lambda : B \rightarrow Q/Z$ depends only on $JB = \text{Ker } i^* \subset J_c(X)$, we obtain the following assertion.

Theorem 2. If the group $\text{Ker } \pi^*$ is Z_2 , where

$$\pi^* : J_c(S^{4n}) \rightarrow J(B) \subset J_c(X),$$

then there is a direct decomposition

$$J(B) = \text{Im } \pi^*(J_c(S^{4n})) + H_J^0(X, Y), \quad H_J^0(X, Y) = J(B)/\pi^*J_c(X, Y).$$

In all cases such a direct decomposition holds for the quotient group B/Z_2 .

II. Let us now dwell in somewhat greater detail on the case when

$$X = S^{4l} \cup_{\varphi} D^{4(k+l)}, \quad k < l.$$

Here, evidently, $Y = S^{4l}$ and $X/Y = S^{4(k+l)}$. For stable two-cell complexes we have the invariant $\bar{\lambda}(\varphi) \in Q/Z$, where for some $\eta \in K_c^0(X)$ we have

$$\text{ch } \eta = a + \bar{\lambda}(\varphi)b;$$

a, b are generators respectively of the groups

$$H^{4l}(X, Z) = H^{4(k+l)}(X, Z) = Z.$$

Let $\alpha \in B \subset K_c^0(X)$. Consider the Thom complex T_ξ for $\xi = \alpha + N$. Then we have, as is easy to show, an element $\mu \in K_c^0(T_\xi)$ such that

$$\text{ch } \mu = A + nB + \delta(n, \xi, \mu)C,$$

where A, B, C are respectively generators of the groups

$$H^{2N}(T_\xi, Z) = H^{2N+4l}(T_\xi, Z) = H^{2N+4(k+l)}(T_\xi, Z) = Z,$$

and $\delta(n, \xi, \mu)$ is a rational number, n an integer. On the other hand, we previously constructed, with the help of ψ_0 , the invariant $\lambda(\xi, \psi_0)$, which defines the mapping $\lambda : B \rightarrow Q/Z$.

There is the simple

Lemma 4. The number $\delta(n, \xi, \mu) \bmod 1$ depends only on n and on ξ .

Using the previously proved Lemma 1, one can easily obtain the following assertion:

Lemma 5. The equality

$$\delta(n, \xi, \mu) = n\bar{\lambda}(\varphi) + \lambda(\xi, \psi_0) \pmod{1}$$

holds.

This lemma follows easily from the fact that, in essence, we defined $\lambda(\xi, \psi_0)$ on B in the same way as $\delta(n, \xi, \mu)$, but taking $n = 0$. The rest follows from this by Lemma 1.

Let B_k be the k -th Bernoulli number and

$$B_k/2k = C_k/D_k,$$

where C_k and D_k are relatively prime.

We indicate the following facts: a) the denominator of $\bar{\lambda}(\varphi)$ is $2D_l$ or a divisor of it; b) the denominator of $\lambda(\xi, \psi_0)$ is $2D_{k+l}$ or a divisor of it.

From this we see that

$$2D_{k+l}(\delta(n, \xi, \mu) - n\bar{\lambda}(\varphi)) \equiv 0 \pmod{1}.$$

Let us now carry out the calculation in the following situation: $\alpha \in K_c(S^{4k})$, $\text{ch } \alpha = 2l + u$, where u is a basis element of the group $H^{4l}(S^{4l}, Z) = Z$, l is a scalar,

$$X = T_\alpha = S^{4l} \cup_\varphi D^{4(k+l)}.$$

We have the Thom isomorphisms

$$\varphi_H : H^j(S^{4k}) \rightarrow H^{4l+j}(T_\alpha), \quad j \geq 0; \quad \varphi_K : K_c(S^{4k}) \rightarrow K_c^0(T_\alpha),$$

where the elements $\varphi_H(1)$ and $\varphi_H(u)$ are denoted by a and b . We choose the isomorphism φ_K so that $\text{ch } \varphi_K(1) = \varphi_H(T(\alpha))$, where $T(\alpha)$ is the Todd genus. Obviously,

$$T(\alpha) = 1 + \frac{B_k}{2k}u.$$

Therefore

$$\text{ch } \varphi_K(1) = a + \frac{B_k}{2k}b = a + \frac{C_k}{D_k}b.$$

It is easy to see that $C_k/D_k = \bar{\lambda}(\varphi) \bmod 1$.

From the results of Adams⁽⁵⁾ we extract that for the element $\bar{\eta} = \varphi_K(1) \in K_c^0(T_\alpha) = K_c^0(X)$, $\eta = 2a_l D_l \bar{\eta}$ lies in $B = \text{Ker } J \cdot i^*$, where $a_l = 1$ if $l = 2l' + 1$; $a_l = 2$ if $l = 2l'$.

Obviously,

$$\text{ch}(2a_l D_l \bar{\eta}) = 2a_l D_l a + \frac{2a_l D_l C_k}{D_k}b.$$

Consider the stable bundle $\xi = N + \eta \in K_c(X)$. Then, as is easy to show,

$$T(\xi) = 1 + (-1)^l 2a_l D_l \left[\frac{C_l}{D_l}a + \frac{C_{k+l}C_k}{D_{k+l}D_k}b \right].$$

For the Thom complex T_ξ of the bundle ξ we shall have

$$\text{ch } \varphi_K(1) = \varphi_{HT}(\xi).$$

Further, from what has been said the following equalities follow:

$$\bar{\lambda}(\varphi) = C_k/D_k, \quad n = (-1)^l 2a_l C_l, \quad \mu = \varphi_K(1),$$

$$\delta(n, \xi, \mu) = (-1)^l 2a_l C_{k+l} C_k D_l / D_{k+l} D_k.$$

Since

$$(\delta(n, \xi, \mu) - n\bar{\lambda}(\varphi))D_{k+l} \equiv 0 \bmod 1,$$

we have

$$2^2 a_l C_k \left(\frac{C_{k+l} D_l - C_l D_{k+l}}{D_k} \right) \equiv 0 \pmod{1}, \quad l > k.$$

Thus the following assertion is obtained:

$$2^2 a_l \frac{C_{k+l} D_l - C_l D_{k+l}}{D_k} \equiv 0 \pmod{1}, \quad l > k,$$

where $B_k/2k = C_k/D_k$ and C_k, D_k are relatively prime.

This relation for Bernoulli numbers was obtained in a strange way from topological considerations.

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