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Abstract

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MATHEMATICS

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APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Let $P_n^{(\alpha, \beta)}(x)$ be the Jacobi polynomials, orthogonal on $[-1, 1]$ with weight $p(x) = (1-x)^\alpha(1+x)^\beta$, normalized by the condition $P_n^{(\alpha, \beta)}(1) = \Gamma(n+\alpha+1)/\Gamma(n+1)\Gamma(\alpha+1)$; $S_n^{(\alpha, \beta)}[f; x]$ is the n -th partial sum of the Fourier series of the function $f(x)$ in the polynomials $P_n^{(\alpha, \beta)}(x)$. As is known,

$$S_n^{(\alpha, \beta)}[f; x] = \int_{-1}^1 f(t) K_n(t, x) p(t) dt,$$

where

$$K_n(t, x) = \lambda_n \frac{P_{n+1}^{(\alpha, \beta)}(t) P_n^{(\alpha, \beta)}(x) - P_n^{(\alpha, \beta)}(t) P_{n+1}^{(\alpha, \beta)}(x)}{t - x},$$

$$\lambda_n = \frac{2^{-\alpha-\beta}}{2n + \alpha + \beta + 2} \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.$$

The aim of the present note is to establish the rate of convergence of $S_n^{(\alpha, \beta)}[f; x]$ to $f(x)$ for various classes of functions.

We consider the following classes of functions, defined on the interval $[-1, 1]$: W^r is the class of functions having an absolutely continuous derivative of order $r-1$ and a derivative of order $r > 0$ satisfying almost everywhere the inequality $|f^{(r)}(x)| \leq 1$; $W^r H_\omega$ is the class of functions having a continuous derivative of order $r > 0$, whose modulus of continuity $\omega(f^{(r)}, \delta) \leq \omega(\delta)$, where $\omega(\delta)$ is a given true majorant of moduli of continuity (i.e. $\lim_{\delta \rightarrow 0} \omega(\delta) = \omega(0) = 0$ and, for $0 \leq \delta_1 < \delta_2$, $0 \leq \omega(\delta_2) - \omega(\delta_1) \leq \omega(\delta_2 - \delta_1)$); $H^\mu = W^0 H_\mu$ is the class of functions satisfying a Lipschitz condition of order μ ($0 \leq \mu \leq 1$) with constant 1; $W^r H^\mu$ is the class $W^r H_\omega$, where $\omega(\delta) = \delta^\mu$.

Theorem 1. Let $\alpha \geq -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$,

$$L_n(x) = \int_{-1}^1 |K_n(t, x)| p(t) dt$$

be the Lebesgue function of the process under consideration. Then the estimate

$$L_n(x) = O(1) \left\{ \ln n + \frac{n^{\alpha+1/2}}{(n\sqrt{1-x})^{\alpha+1/2} + 1} + \frac{n^{\beta+1/2}}{(n\sqrt{1+x})^{\beta+1/2} + 1} \right\}$$

is valid. Here $x \in [-1, 1]$, and the quantity $O(1)$ depends only on α and β .

Remark 1. The order of the indicated estimate cannot be improved, since, as Rau showed ⁽¹⁾, $L_n(1) \sim n^{\alpha+1/2}$.

Remark 2. Using the well-known inequality

$$|S_n^{(\alpha, \beta)}[f; x] - f(x)| = [L_n(x) + 1]E_n(f),$$

where $E_n(f)$ is the best approximation of the function $f(x)$ by algebraic polynomials of degree not exceeding n , one can find estimates for the deviation of the Fourier-Jacobi sums from the approximated functions for all possible classes of functions.

* The class H^0 consists of measurable functions f for which $|f(x) - f(y)| \leq 1$ for all x and y .

Theorem 2. The asymptotic formula holds

$$\begin{aligned} & \sup_{f \in H^\mu} |S_n^{(\alpha, \beta)}[f; 1] - f(1)| = \\ &= \frac{2^{2\mu}}{\pi^{1/2}} \int_0^{\pi/2} u^\mu \sin u \, du \frac{\Gamma(\mu/2 + \alpha/2 + 1/4) \Gamma(\mu/2 + \beta/2 + 3/4)}{\Gamma(\alpha + 1) \Gamma(\mu + \alpha/2 + \beta/2 + 1)} n^{\alpha+1/2-\mu} + \sigma_n n^{-\mu}. \end{aligned}$$

Here $\alpha > -1/2$, $\beta > -1/2$,

$$\sigma_n = \begin{cases} O(n^{\alpha-1/2} \ln n), & \text{if } \alpha = 1/2, \beta \geq -1/2 \text{ or } \alpha \geq 1/2, \beta = -1/2, \\ O(n^{\alpha-1/2}) + O(1) + O(n^{\alpha-\beta-1}) & \text{in the remaining cases;} \end{cases}$$

the constants entering the O -terms depend only on α and β .

Remark 1. If $-1 < \alpha \leq -1/2$, then

$$\sup_{f \in H^\mu} |S_n^{(\alpha, \beta)}[f; 1] - f(1)| = O(n^{-\mu}).$$

Remark 2. Since

$$\sup_{f \in H^0} |S_n^{(\alpha, \beta)}[f; 1] - f(1)| = \frac{1}{2} L_n(1),$$

from Theorem 2, for $\mu = 0$, one obtains the asymptotic formula for $L_n(1)$ found by Lorch ⁽²⁾.

Theorem 3. If the function $f(x) \in W^r$, then

$$\begin{aligned} S_n^{(\alpha, \beta)}[f; 1] - f(1) &= A \{S_{n-r}^{(\alpha+r, \beta+r)}[f^{(r)}; 1] - f^{(r)}(1)\} + \\ &+ B \int_{-1}^1 g(t) P_{n-r-1}^{(\alpha+r+2, \beta+r+1)}(t) (1-t)^{\alpha+r} (1+t)^{\beta+r+1} dt, \end{aligned} \quad (*)$$

where $\alpha > -1$, $\beta > -1$,

$$\begin{aligned} A &= 2^r \frac{\Gamma(\alpha+r+1)\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\alpha+\beta+r+2)} \frac{(n-r)!}{n!}, \\ B &= 2^{-\alpha-\beta-r-2} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} \frac{(n-r-1)!}{n!}, \end{aligned}$$

$$g(t) = f^{(r)}(1) + r f^{(r)}(t) - r(r+1) \int_0^1 f^{(r)}(1-u+tu) u^{r-1} du.$$

Remark 1. If $f(x) \in W^r H_\omega$, then the second term on the right-hand side of formula (*) is

$$O(\omega(n^{-1})(n^{\alpha-r-1/2} + n^{-2r})),$$

when $\alpha + r \neq 1/2$, and

$$O(\omega(n^{-1})n^{2r} \ln n),$$

when $\alpha + r = 1/2$. Here the constants entering the O -terms depend on α, β , and r .

Remark 2. From Theorems 2 and 3 and the preceding remark there easily follows the asymptotic formula for

$$\sup_{f \in W^r H^\mu} \left| S_n^{(\alpha, \beta)}[f; 1] - f(1) \right|.$$

Theorem 4. Let $\alpha > -1$, $\beta > -1$. Then for $x \in (-1, 1)$ and $r \geq 0$ one has

$$\begin{aligned} & \sup_{f \in W^r H^\mu} \left| S_n^{(\alpha, \beta)}[f; x] - f(x) \right| = \\ &= \frac{2^{\mu+1}}{\pi^2} \int_0^{\pi/2} u^\mu \sin u \, du \left(\frac{\sqrt{1-x^2}}{n} \right)^{r+\mu} \ln n + O(n^{-r-\mu}), \end{aligned}$$

where the remainder term depends on α, β, r , and x . The estimate of the remainder term is uniform with respect to x on the interval $[-1 + \varepsilon, 1 - \varepsilon]$.

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Note: Figure translations are in progress. See original paper for figures.

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