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A. V. Ivanov, O. A. Ladyzhenskaya, A. L. Treskunov, N. N.
Ural'tseva

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Abstract

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MATHEMATICS

A. V. IVANOV, O. A. LADYZHENSKAYA, A. L. TRESKUNOV, N. N. URAL'
TSEVA

SOME PROPERTIES OF GENERALIZED SOLUTIONS OF SECOND-ORDER PARABOLIC EQUATIONS

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The paper investigates generalized solutions (g.s.) of linear and quasilinear uniformly parabolic second-order equations, to which the papers ⁽¹⁻⁶⁾ are devoted. Here, to characterize the coefficients of the equation, a continuous scale of norms of the spaces $L_{p,p'}(Q_T)$ is used, i.e. the norm

$$\|u\|_{p,p_1,Q_T} = \left(\int_0^T \left(\int_{\Omega} |u|^p dx \right)^{p_1/p} dt \right)^{1/p_1}, \quad (1)$$

where Ω is some bounded domain contained in the Euclidean space E_n , and $Q_T = \Omega \times [0, T]$.

Let $Q = \Omega \times [t_1, t_2]$. By $V_2^{1,0}(Q)$ ($\dot{V}_2^{1,0}(Q)$) we denote the closure of all functions smooth in $\bar{Q}$ (all smooth functions equal to zero near the lateral surface of Q) in the norm

$$u\|_{V_2^{1,0}(Q)} = \left(\max_{t \in [t_1, t_2]} \int_{\Omega} u^2(x, t) dx + \iint_Q |\text{grad } u|^2 dx dt \right)^{1/2}. \quad (2)$$

Denote by $K_{\rho} = K_{\rho}(x_0)$ the ball of radius ρ with center at the point x_0 contained in E_n ; by $Q_{\rho,\tau} = Q_{\rho,\tau}(x_0, t_0)$ the cylinder $K_{\rho}(x_0) \times [t_0 - \tau, t_0]$, and let $v_+ = \max(v, 0)$.

Lemma 1. Let the numbers $r > 0$, $k' \geq 0$ be fixed, and let the function $v(x, t)$ belong to $V_2^{1,0}(Q_{r,r^2})$ and, for all $k \geq k'$, $\sigma \in (0, 1)$, $\theta \in (0, 1)$, $\rho \in [r/2, r]$, $\tau \in [(r/2)^2, r^2]$, satisfy the inequality

$$\max_{t \in [t_0 - \tau, t_0]} r^{-n} \int_{K_{\rho - \sigma \rho}, v(t) > k} (v - k)^2 dx + r^{-n} \iint_{(Q_{\rho - \sigma \rho, \tau - \theta \tau}, v > k)} |\text{grad } v|^2 dx dt \leq$$

$$\begin{aligned} &\leq \gamma_0(\sigma^{-2} + \theta^{-1})r^{-(n+2)} \iint_{(Q_{\rho,\tau}, v>k)} (v-k)^2 dx dt + \\ &+ \gamma_1 k^\delta \left\{ r^{-2} \int_{t_0-\tau}^{t_0} \left[\frac{\text{mes}(K_\rho, v(t) > k)}{\text{mes } K_r} \right]^{\alpha-2/n} dt \right\}^\beta, \end{aligned} \quad (3)$$

where $\gamma_0 \geq 0$, $\gamma_1 \geq 0$, $0 \leq \delta \leq 2$, $1 \leq \alpha \leq \infty$, $\beta \geq 0$, $\alpha\beta = 1 + \varepsilon$, $\varepsilon > -1$.*

* We agree that for $\alpha = \infty$ the expression

$$\left\{ r^{-2} \int_{t_0-\tau}^{t_0} [\text{mes}(K_\rho, v(t) > k) \text{mes}^{-1} K_r]^{\alpha-2/n} dt \right\}^{(1+\varepsilon)/\alpha}$$

is understood as the limit of the written expression as $\alpha \rightarrow \infty$, i.e. as

$$\max_{t \in [t_0-\tau, t_0]} [\text{mes}(K_\rho, v(t) > k) \cdot \text{mes}^{-1} K_r]^{1+\varepsilon}.$$

1) Let $-1 < \varepsilon < 0$, $\delta = 0$. Then

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_1 h^{-2(n+2)/n|\varepsilon|} \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right)^{2(n+2)/n|\varepsilon|}, \\ h &\geq h_0 = c_0^{(1)} \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right). \end{aligned} \quad (4)$$

2) Let $\varepsilon = 0$, $0 \leq \delta \leq 2$, and let p be any number greater than 1. There exists a number d , depending only on n , such that for $\gamma_1 \leq de^{-p}$ the inequality holds

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_2 h^{-p} \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right)^p, \\ h &\geq h_0 = c_0^{(2)} \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right). \end{aligned} \quad (5)$$

3) Let $\varepsilon = 0$, $\delta = 0$. Then

$$\begin{aligned} r^{-(n+2)} \text{mes}(Q_{r/2, r^2/2}, v > h) &\leq c_3 \exp \left\{ -c_4 \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right)^{-1} h \right\}, \\ h &\geq h_0 = c_0^{(3)} \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right). \end{aligned} \quad (6)$$

4) Let $\varepsilon > 0$, $0 \leq \delta \leq 2$. Then

$$\text{vrai max}_{Q_{r/2, r^2/2}} v \leq c_5 \left(r^{-(n+2)/2} \|v_+\|_{2,2,Q_{r,r^2}} + 1 \right). \quad (7)$$

In inequalities (4)–(7), the constants $c_0^{(i)}$ ($i \neq 2$) and c_j ($j \neq 2$) depend only on $n, \gamma_0, \gamma_1, k', \alpha$, and β , while the constants $c_0^{(2)}, c_2$ depend on the same quantities and on the number p .

For $\alpha = 1 + 2/n$, Lemma 1 was proved in ⁽⁶⁾.

Consider in Q_T the linear equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial u}{\partial x_j} + A_i u \right) + B_i \frac{\partial u}{\partial x_i} + Cu + \frac{\partial}{\partial x_i} F_i + G = 0 \quad (8)$$

under the condition that

$$\nu \sum_{i=1}^n \xi_i^2 \leq A_{ij}(x, t) \xi_i \xi_j \leq \nu^{-1} \sum_{i=1}^n \xi_i^2,$$

where $\nu > 0$. Let

$$\begin{aligned} \vec{A} &\in L_{a,a_1}(Q_T), & \vec{B} &\in L_{b,b_1}(Q_T), & C &\in L_{c,c_1}(Q_T), & \vec{F} &\in L_{f,f_1}(Q_T), \\ & & & & G &\in L_{g,g_1}(Q_T), \end{aligned}$$

and moreover

$$\|\vec{A}\|_{a,a_1,Q_T} + \|\vec{B}\|_{b,b_1,Q_T} + \|C\|_{c,c_1,Q_T} \leq \mu.$$

A generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$ ($\dot{V}_2^{1,0}(Q_T)$) is a function $u(x, t)$ belonging to $V_2^{1,0}(Q_T)$ ($\dot{V}_2^{1,0}(Q_T)$) and satisfying, for all $t_1, t_2 \in [0, T]$ and all $\Phi(x, t) \in \dot{W}_2^1(\Omega \times [t_1, t_2])$, the identity

$$\begin{aligned} &\int_{\Omega} u \Phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \left[-u \frac{\partial \Phi}{\partial t} + \left(A_{ij} \frac{\partial u}{\partial x_j} + A_i u \right) \frac{\partial \Phi}{\partial x_i} + \right. \\ &\quad \left. + \left(B_i \frac{\partial u}{\partial x_i} + Cu \right) \Phi + F_i \frac{\partial \Phi}{\partial x_j} + G \Phi \right] dx dt = 0. \end{aligned}$$

Theorem 1. Let

$$\begin{aligned} 1/a + 2/na_1 &= 1/n, & a_1 &\in [2, \infty), & n &\geq 2; \\ 1/b + 2/nb_1 &= 1/n, & b_1 &\in [2, \infty), & n &\geq 2; \\ 1/c + 2/nc_1 &= 2/n, & c_1 &\in [1, \infty), & n &\geq 2; \\ f &= 2, & f_1 &= 2; \\ 1/g + 2/ng_1 &= 2/n + \frac{1}{2}, & g_1 &\in [1, 2] \text{ for } n > 2, & g_1 &\in [1, 2] \text{ for } n = 2. \end{aligned}$$

Then every generalized solution of equation (8) from the class $\dot{V}_2^{1,0}(Q_T)$ satisfies the inequality

$$\|u\|_{V_2^{1,0}(Q_T)} \leq c \left[\|\vec{F}\|_{f,f_1,Q_T} + \|G\|_{g,g_1,Q_T} + \|u(x, 0)\|_{2,\Omega} \right],$$

where

$$c = c(n, \nu, \mu, a_1, b_1, c_1).$$

Theorem 2. Under the assumptions of Theorem 1 and for any $\varphi(x) \in L_2(\Omega)$, there exists a unique generalized solution of equation (8) from the class $\overset{\circ}{V}_2^{1,0}(Q)$, equal to $\varphi(x)$ for $t = 0$. Every generalized solution from this class satisfies the condition

$$\|u(x, t+h) - u(x, t)\|_{2,2,Q_T} = o(h^{1/2}).$$

Denote by Γ the set of points lying on the lateral surface and the lower base of the cylinder Q_T . Let Q' denote an arbitrary domain contained in Q_T and at positive distance from Γ .

Theorem 3. Let

$$1/a + 2/na_1 = 1/n, \quad a_1 \in [2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n, \quad c_1 \in [1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n + \sigma/2, \quad \sigma \in (0, 1), \quad f_1 \in [2, 2/\sigma] \text{ for } n > 2, \quad f_1 \in [2, 2/\sigma] \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n + \theta/2, \quad \theta \in (0, 1), \quad g_1 \in [1, 2/\theta] \text{ for } n > 2, \quad g_1 \in [1, 2/\theta] \text{ for } n = 2.$$

Then an arbitrary generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$ belongs to the space $L_{\chi, \chi_1}(Q')$, where

$$1/\chi + 2/n\chi_1 = \lambda/2, \quad \chi_1 \in [2/\lambda, \infty] \text{ for } n > 2, \quad \chi_1 \in (2/\lambda, \infty] \text{ for } n = 2, \quad \lambda = \max(\sigma, \theta).$$

If this generalized solution is bounded on Γ , then it also belongs to $L_{\chi, \chi_1}(Q_T)$ with the same exponents χ and χ_1 .

Theorem 4. Suppose that, in the assumptions of Theorem 3, the exponents σ and θ are equal to zero.

Then an arbitrary generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$ belongs to the space $L_{\chi, \chi_1}(Q')$ with arbitrary χ and χ_1 . If this generalized solution is bounded on Γ , then it also belongs to $L_{\chi, \chi_1}(Q_T)$.

Theorem 5. Let

$$1/a + 2/na_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad a_1 \in (2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n - \delta, \quad \delta > 0, \quad c_1 \in (1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n, \quad f_1 \in [2, \infty] \text{ for } n > 2, \quad f_1 \in [2, \infty) \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n, \quad g_1 \in [1, \infty] \text{ for } n > 2, \quad g_1 \in [1, \infty) \text{ for } n = 2.$$

Then there exists a constant d , depending only on n, ν, μ , such that the integrals

$$E_{Q'}(u) = \iint_{Q'} \exp\{d(\|u\|_{2,2,Q_T} + 1)^{-1}|u(x,t)|\} dx dt$$

are finite for any generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$. If this solution is bounded on Γ , then the integral $E_{Q_T}(u)$ is also finite.

Theorem 6. Let

$$1/a + 2/na_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad a_1 \in (2, \infty), \quad n \geq 2;$$

$$1/b + 2/nb_1 = 1/n, \quad b_1 \in [2, \infty), \quad n \geq 2;$$

$$1/c + 2/nc_1 = 2/n + \delta, \quad \delta > 0, \quad c_1 \in (1, \infty), \quad n \geq 2;$$

$$1/f + 2/nf_1 = 1/n - \sigma, \quad \sigma > 0, \quad f_1 \in (2, \infty] \text{ for } n > 2, \quad f_1 \in (2, \infty) \text{ for } n = 2;$$

$$1/g + 2/ng_1 = 2/n - \theta, \quad \theta > 0, \quad g_1 \in (1, \infty] \text{ for } n > 2, \quad g_1 \in (1, \infty) \text{ for } n = 2.$$

Then every generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$ is bounded in every domain Q' . If this generalized solution is bounded on Γ , then it will also be bounded in the whole cylinder Q_T .

Theorem 7. Under the assumptions of Theorem 6, an arbitrary generalized solution of equation (8) from the class $V_2^{1,0}(Q_T)$ is Hölder continuous in any domain Q' . If, in addition, such a generalized solution is Hölder continuous on Γ , then it will have a bounded Hölder constant in the whole cylinder Q_T .

We now consider the quasilinear equation

$$\frac{\partial u}{\partial t} - \frac{d}{dx_i} (A_i(x, t, u, u_x)) + A(x, t, u, u_x) = 0, \quad (9)$$

in which the functions $A_i(x, t, u, \vec{p})$ and $A(x, t, u, \vec{p})$ are defined for $(x, t) \in Q_T$, $|u| \leq M$, and arbitrary $p = (p_1, \dots, p_n)$, are continuous in u and p_k , and satisfy the conditions

$$A_i p_i \geq \nu |\vec{p}|^2 - \varphi^{(1)}, \quad |A_i| \leq \mu |\vec{p}| + \varphi^{(2)}, \quad |A| \leq \mu |\vec{p}|^2 + \varphi^{(3)}, \quad (10)$$

where $\nu, \mu > 0$, $\varphi^{(1)} \in L_{q,q_1}(Q_T)$, $\varphi^{(2)} \in L_{r,r_1}(Q_T)$, $\varphi^{(3)} \in L_{s,s_1}(Q_T)$, $\varphi^{(i)} \geq 0$, $i = 1, 2, 3$.

Theorem 8. Let $1/q + 2/nq_1 = 2/n - \varepsilon$, $\varepsilon > 0$, $q_1 \in (1, \infty]$ for $n > 2$, $q_1 \in (1, \infty)$ for $n = 2$; $1/r + 2/nr_1 \leq 1/n - \delta$, $\delta > 0$, $r_1 \in (2, \infty]$ for $n > 2$, $r_1 \in (2, \infty)$ for $n = 2$; $1/s + 2/ns_1 = 2/n - \chi$, $\chi > 0$, $s_1 \in (1, \infty]$, $n > 2$, $s_1 \in (1, \infty)$, $n = 2$.

Then any bounded generalized solution from the class $V_2^{1,0}(Q_T)$ of equation (9)*, satisfying conditions (10), belongs to the Hölder space $C_{x,t}^{\alpha,\alpha/2}(Q')$ with some $\alpha > 0$ for any domain $Q' \subset \overline{Q'} \subset Q_T$.

Finally, let $u(x, t)$ be a solution of the equation

$$\frac{\partial u}{\partial t} - A_{ij}(x, t, u, u_x)u_{x_i x_j} + A(x, t, u, u_x) = 0,$$

in which the functions $A_{ij}(x, t, u, \vec{p})$, $A(x, t, u, \vec{p})$ are defined in a neighborhood of the manifold

$$\mathfrak{M} = \{(x, t) \in \overline{Q_T}, u = u(x, t), \vec{p} = u_x(x, t)\}$$

and are continuous in u and p . Suppose further that $A_{ij}(x, t, u, p)$ are continuously differentiable with respect to x, u , and p_k , and that on the solution $u(x, t)$ the functions A_{ij} and A satisfy the conditions

$$\nu \sum_{i=1}^n \xi_i^2 \leq A_{ij}(x, t) \xi_i \xi_j \leq \nu^{-1} \sum_{i=1}^n \xi_i^2, \quad \nu > 0,$$

$$|\partial A_{ij}/\partial p_k| \leq \mu, \quad |\partial A_{ij}/\partial u; \partial A_{ij}/\partial x_k, A| \leq \varphi(x, t), \quad \varphi(x, t) \geq 0,$$

$$\varphi \in L_{q, q_1}(Q_T), \quad 1/q + 2/nq_1 = 1/n - \varepsilon, \quad \varepsilon > 0, \quad q_1 \in (2, \infty), \quad n \geq 2.$$

Theorem 9. Under the assumptions made above, the Hölder norm

$$|u_x|_{C_{x,t}^{\alpha, \alpha/2}(Q')}$$

with some $\alpha > 0$ is estimated solely in terms of n, ν, μ, q, q_1 , the distance from Q' to Γ , and

$$\max_{Q_T} |u_x|.$$

Leningrad Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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* That is, any function $u(x, t)$ belonging to $V_2^{1,0}(Q_T)$, satisfying for all $t_1, t_2 \in [0, T]$ and bounded $\Phi(x, t) \in W_2^1(\Omega \times [0, T])$ the integral identity

$$\int_{\Omega} u \Phi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} [A_i \Phi_{x_i} + A \Phi] dx dt = 0$$

and such that $|u(x, t)| \leq M$.

Note: Figure translations are in progress. See original paper for figures.

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