

ON CLOSED IDEALS IN A LOCALLY CONVEX ALGEBRA OF ENTIRE FUNCTIONS WITH AN ARBITRARY GROWTH MAJORANT

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Abstract

Full Text

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MATHEMATICS

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ON CLOSED IDEALS IN A LOCALLY CONVEX ALGEBRA OF ENTIRE FUNCTIONS WITH AN ARBITRARY GROWTH MAJORANT

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Let $M(t)$ be a monotonically increasing positive function of $t \geq 0$, satisfying the condition: for sufficiently large t , $\ln \ln M(t)$ is convex in $\ln t$. From this condition, in particular, there follows the following property of the function $M(t)$: for any positive numbers a, b there exist numbers $C = C(a, b)$ and $c = c(a, b)$ such that for all $t \geq 0$ the inequality

$$M(at)M(bt) \leq CM(ct). \quad (*)$$

holds.

Define the class P of entire functions as follows: $\varphi \in P$ if and only if there exist numbers $a = a(\varphi)$, $A = A(\varphi)$ such that, for all z ,

$$|\varphi(z)| \leq AM(a|z|).$$

If $\varphi_1, \varphi_2 \in P$, then, by virtue of $(*)$, also $\varphi_1\varphi_2 \in P$. Therefore, with the usual operations of addition and multiplication, P is an algebra over the field of complex numbers. By the letter m we shall denote any monotonically increasing positive function $m(t)$ of the argument $t \geq 0$. Let

$$\mathfrak{B} = \left\{ m : \lim_{t \rightarrow \infty} \frac{m(t)}{M(at)} = \infty \text{ for any } a > 0 \right\}.$$

Introduce in P a locally convex topology by taking, as a fundamental system of neighborhoods of zero (see ⁽¹⁾, p. 367), the system of all absolutely convex and absorbing sets of the form

$$V_m = \{ \varphi \in P : |\varphi(z)| \leq m(|z|) \text{ for all } z \}, \quad m \in \mathfrak{B}.$$

With this topology P is a semi-reflexive space. Convergence of a countable sequence $\varphi_n \rightarrow \varphi$ in the topology of P is equivalent to the following:

- 1) $\varphi_n(z) \rightarrow \varphi(z)$ uniformly on each compact set in the complex plane;
- 2) there exist numbers $A = A(\{\varphi_n\})$, $a = a(\{\varphi_n\})$ such that, for all z and n , the inequalities

$$|\varphi_n(z) - \varphi(z)| \leq AM(a|z|)$$

hold.

These features of the space P indicate that the topology under consideration is natural for the class P .

The operation of pointwise multiplication of functions turns out to be continuous in this topology. Hence P is a locally convex algebra. This algebra, however, is not countably normed. Using the terminology of article (2), we shall say that an ideal $I \subset P$ is fixed if there exists at least one point λ such that $f(\lambda) = 0$ for every $f \in I$. The point λ will be called a zero of the ideal I . The multiplicity of a zero of an ideal is defined in the natural way.

For the algebra P the following is valid.

Theorem on closed ideals. *Every nontrivial closed ideal $I \subset P$ is fixed. If λ_i ($i = 1, \dots$) is the sequence of all its zeros, then for some $a > 0$*

$$\overline{\lim}_{t \rightarrow \infty} \frac{n(t)}{\ln M(at)} < \infty, \quad (**)$$

where $n(t)$ is the number of zeros λ_i , counted with their multiplicities, in the disk $\{z : |z| \leq t\}$. Conversely, if λ_i ($i = 1, \dots$) is an arbitrary sequence of points satisfying, for some $a > 0$, condition (**), then there exists a nontrivial closed ideal $I \subset P$ whose zeros are the points λ_i and only these points. The ideal I is uniquely determined by its zeros.

Thus, the closed ideals in P are completely characterized by their zeros. The theorem formulated above is a generalization of the result of Ehrenpreis (3), which pertains only to the algebra of entire functions of exponential type: $M(t) = e^t$. It is also useful to compare it with the results of F. Schilling (4) and P. K. Rashedskii (5). The proof uses the notions of a generalized shift of entire functions and a generalized convolution of functionals in the space of all entire functions. In this setting the algebra P turns out to be isomorphic to an algebra of linear functionals with convolution. At the final stage of the proof an infinite product is constructed representing a function from P . The method for constructing such a product was indicated by the author earlier (see (6), p. 175). Here an essential role is played by the condition that $\ln \ln M(t)$ is convex in $\ln t$. In conclusion, we note that the notion of generalized shift is closely connected with the theory of analytic operators of A. F. Leont'ev (see (7), p. 19).

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CITED LITERATURE

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