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Abstract

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MATHEMATICS

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ON ONE CASE OF A DISCONTINUOUS RIEMANN PROBLEM

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Let $t = t(s)$ be the equation of a simple rectifiable curve C , where s is arc length. By L_p , $p \geq 1$, we denote the space of complex-valued functions $\varphi(t)$ on C for which $\|\varphi(t)\|_p = \left(\int_C |\varphi(t(s))|^p ds \right)^{1/p} < \infty$. Let $\omega_p(\delta, \varphi) = \max_{|h| < \delta} \|\varphi(t(s+h)) - \varphi(t(s))\|_p$ be the integral modulus of continuity of the function $\varphi(t)$. We shall say that $\varphi(t) \in H_p^\alpha$ if $\omega_p(\delta, \varphi) \leq A\delta^\alpha$, where A, α are certain constants, $0 < \alpha \leq 1$. We shall call C a curve of class K if there exists a constant $k > 0$ such that for any points $t_1, t_2 \in C$ we have

$$|t_1 - t_2| \geq k\sigma(t_1, t_2), \quad (1)$$

where $\sigma(t_1, t_2)$ is the length of the smallest arc with endpoints at t_1 and t_2 . By $H(\mu)$ we denote the class of functions satisfying on C the Hölder condition with exponent μ , $0 < \mu \leq 1$.

1°. Theorem 1. Let $\varphi(t) \in H_{p_1}^{\alpha_1}$, $\psi(t) \in H_{p_2}^{\alpha_2}$ on a rectifiable curve C , $1/p_1 + 1/p_2 \leq 1$, $0 < \alpha_1, \alpha_2 \leq 1$. Then $\varphi(t)\psi(t) \in H_{p_3}^{\alpha_3}$, where $p_3 = p_1 p_2 / (p_1 + p_2)$, $\alpha_3 = \min(\alpha_1, \alpha_2)$. (If $\psi(t) \in H(\alpha_2)$, then $\varphi(t)\psi(t) \in H_{p_1}^{\alpha_3}$.)

Theorem 2. Let C be a simple closed curve of class K , $\varphi(t) \in H_{p_1}^{\alpha_1}$, $\psi(t) \in H_{p_2}^{\alpha_2}$ on C , where $1/p_1 + 1/p_2 < 1$, $0 < \alpha_1, \alpha_2 \leq 1$. Then almost everywhere on C the following equality holds (the Poincaré-Bertrand formula):

$$\int_C \frac{\varphi(t)}{t - t_0} dt \int_C \frac{\psi(\tau)}{\tau - t} d\tau = -\pi^2 \varphi(t_0) \psi(t_0) + \int_C \psi(\tau) d\tau \int_C \frac{\varphi(t) dt}{(t - t_0)(\tau - t)}. \quad (2)$$

The existence of each of the repeated integrals in equality (2) follows from Theorem 1 and from the fact that

$$\int_C \frac{g(t) dt}{t - \tau} \in H_p^\alpha, \quad \text{if } C \in K \text{ and } g(t) \in H_p^\alpha.$$

The latter was proved in paper (1).

Formula (2) is obtained from the equality

$$\int_C \frac{\varphi(t)}{t-z} dt \int_C \frac{\psi(\tau)}{\tau-t} d\tau = \int_C \psi(\tau) d\tau \int_C \frac{\varphi(t) dt}{(t-z)(\tau-t)}, \quad z \notin C, \quad (3)$$

by passing to the limit as $z \rightarrow t_0$, using the known Sokhotski formulas, which in the case under consideration are valid (see (1)). To prove equality (3), as in the case considered in (2), it is sufficient to show that

$$\lim_{\varepsilon \rightarrow 0} \int_C \frac{\varphi(t) dt(s)}{t-z} \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(\tau) d\tau(\sigma)}{\tau-t} = 0, \quad \lim_{\varepsilon \rightarrow 0} \int_C \psi(\tau) d\tau(\sigma) \int_{\sigma-\varepsilon}^{\sigma+\varepsilon} \frac{\varphi(t) dt(s)}{(t-z)(\tau-t)} = 0. \quad (4)$$

Here $t = t(s)$, $\tau = \tau(\sigma)$; s, σ are the arc abscissae of the points t and τ . We shall prove

the first of these equalities. We represent the integral entering into it in the form of a sum

$$\begin{aligned} \int_C \frac{\varphi(t) dt(s)}{t-z} \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(\tau) d\tau(\sigma)}{\tau-t} &= \int_C \frac{\varphi(t) dt(s)}{t-z} \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(\tau) - \psi(t)}{\tau-t} d\tau(\sigma) + \\ &+ \int_C \frac{\varphi(t) \psi(t) dt(s)}{t-z} \int_{s-\varepsilon}^{s+\varepsilon} \frac{d\tau(\sigma)}{\tau-t} = I_1 + I_2. \end{aligned}$$

Considering z fixed, put $m = \min_{t \in C} |t-z|$. Then

$$|I_1| \leq \frac{\|\varphi(t)\|_r}{m} \left(\int_C \left| \int_{s-\varepsilon}^{s+\varepsilon} \frac{\psi(\tau(\sigma)) - \psi(t(s))}{\tau(\sigma) - t(s)} \tau'(\sigma) d\tau \right|^{p_2} ds \right)^{1/p_2},$$

where $r = p_2/(p_2-1) < p_1$. Putting $\sigma = s_1 + s$, on the basis of the generalized Minkowski inequality we obtain

$$|I_1| \leq M_1 \int_{-\varepsilon}^{\varepsilon} \left(\int_C \frac{|\psi(t(s_1+s)) - \psi(t(s))|^{p_2}}{|t(s_1+s) - t(s)|^{p_2}} ds \right)^{1/p_2} ds_1, \quad M_1 = \text{const.}$$

By virtue of (1), $|t(s_1+s) - t(s)| \geq k|s_1|$. Therefore

$$|I_1| \leq \frac{2M_1}{k} \int_0^\varepsilon \frac{\omega_{p_2}(\delta, \psi)}{\delta} d\delta \leq \frac{2AM_1}{k\alpha_2} \varepsilon^{\alpha_2}.$$

Hence $\lim_{\varepsilon \rightarrow 0} I_1 = 0$.

Now consider the integral I_2 . Almost everywhere on C we have

$$\int_{s-\varepsilon}^{s+\varepsilon} \frac{d\tau(\sigma)}{\tau - t} = \int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{d\tau}{\tau - t} = \ln \left| \frac{t(s+\varepsilon) - t(s)}{t(s-\varepsilon) - t(s)} \right| + i\omega, \quad (5)$$

where ω is the angle between the vectors $t(s+\varepsilon) - t(s)$ and $t(s) - t(s-\varepsilon)$, measured from the latter and satisfying the inequalities $-\pi < \omega < \pi$.

Since almost everywhere on C

$$\lim_{\varepsilon \rightarrow 0} \left| \frac{t(s+\varepsilon) - t(s)}{t(s-\varepsilon) - t(s)} \right| = 1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \omega = 0,$$

it follows from (5) that

$$\lim_{\varepsilon \rightarrow 0} \int_{t(s-\varepsilon)}^{t(s+\varepsilon)} \frac{d\tau}{\tau - t} = 0 \quad (6)$$

for almost all t on C . On the basis of (1), for all s and $n = 1, 2, \dots$ we have

$$0 < k \leq \frac{|t(s+1/n) - t(s)|}{|t(s-1/n) - t(s)|} \leq \frac{1}{k}.$$

Therefore, almost everywhere on C and for any n ,

$$\left| \int_{t(s-1/n)}^{t(s+1/n)} \frac{d\tau}{\tau - t} \right| < \sqrt{\ln^2 k + \pi^2} = M_2 = \text{const}. \quad (7)$$

Returning to the integral I_2 , we note that

$$\lim_{\varepsilon \rightarrow 0} |I_2| \leq \lim_{n \rightarrow \infty} \frac{\|\varphi(t)\psi(t)\|_{p_3}}{m} \left(\int_C \left| \int_{s-1/n}^{s+1/n} \frac{d\tau(\sigma)}{\tau - t} \right|^{p_3/(p_3-1)} \right)^{(p_3-1)/p_3}.$$

Denote

$$f_n(s) = \left| \int_{s-1/n}^{s+1/n} \frac{d\tau(\sigma)}{\tau - t} \right|^{p_3/(p_3-1)}, \quad n = 1, 2, \dots$$

By virtue of (7),

$$f_n(s) < M_2^{p_3/(p_3-1)}$$

almost everywhere on C and for all n simultaneously. Moreover, on the basis of (6), almost everywhere on C we have $\lim_{n \rightarrow \infty} f_n(s) = 0$. Then

$$\lim_{n \rightarrow \infty} \int_C f_n(s) ds = \int_C \lim_{n \rightarrow \infty} f_n(s) ds = 0$$

and $\lim_{\varepsilon \rightarrow 0} I_2 = 0$. It was shown earlier that $\lim_{\varepsilon \rightarrow 0} I_1 = 0$. Thus, the first of the equalities (4) is proved. The second is proved analogously.

Let $I(C, B, P(z))$ denote the class of functions representable by a Cauchy-type integral with density from some class B , with line of jumps C and principal part $P(z)$ at infinity.

Theorem 3. Let $\Phi(z) \in I(C, H_{p_1}^{\alpha_1}, P(z))$, $\Psi(z) \in I(C, H_{p_2}^{\alpha_2}, Q(z))$ ($\Psi(z) \in I(C, H(\alpha_2), Q(z))$), where C is a simple closed curve of class K , $1/p_1 + 1/p_2 < 1$. Then

$$\begin{aligned} \Phi(z)\Psi(z) &\in I(C, H_{p_3}^{\alpha_3}, R(z)) \\ (\Phi(z)\Psi(z) &\in I(C, H_{p_1}^{\alpha_3}, R(z))), \quad p_3 = \frac{p_1 p_2}{p_1 + p_2}, \quad \alpha_3 = \min(\alpha_1, \alpha_2). \end{aligned}$$

This assertion follows from Theorem 2, and also from Theorem 1.10.1 in ⁽⁴⁾. We note that if the orders at infinity of the functions $\Phi(z)$ and $\Psi(z)$ are respectively $n \geq 0$ and $m \geq 0$, then $R(z)$ is a polynomial of degree $n + m$. In the case when $n < 0$, but $n + m \geq 0$, $R(z)$ is a polynomial of degree not higher than $n + m$.

2°. Let C be a simple closed curve of class K , bounding a finite domain D^+ . Denote by D^- the complement of $D^+ + C$ to the full plane. We assume that the point $z = 0$ is contained in D^+ . Consider the following Riemann problem: to find a piecewise-analytic function $\Phi(z) = \Phi^\pm(z)$, belonging to the class $I(C, H_p^\alpha, a_0)$, $a_0 = \text{const}$, such that its angular boundary values $\Phi^\pm(t)$ almost everywhere on C satisfy the equality

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad (8)$$

where $G(t) \in H(\alpha)$, $G(t) \neq 0$, $g(t) \in H_p^\alpha$ on C , $p \geq 1$, $0 < \alpha \leq 1$.

This problem was studied in ^(2,3) in the case when C is a smooth curve and $G(t), g(t) \in H(\alpha)$. In ⁽⁴⁻⁹⁾ the requirements imposed on $G(t), g(t)$ are considerably weakened, but additional restrictions are imposed on the curve C . In ⁽¹⁰⁻¹²⁾, conversely, the class of curves C is enlarged, but the functions $G(t), g(t)$

remain the same as in ^(2,3). In the present work, in comparison with ^(2,3), both the class of contours and the class of coefficients of the boundary condition are enlarged simultaneously.

1) Let $\text{ind } G(t) = \nu \geq 0$. Relying on (13), it is easy to show that

$$t^{-\nu} G(t) = X^+(t)[X^-(t)]^{-1},$$

where $X^\pm(t)$ are the boundary values of the piecewise-analytic function

$$X^\pm(z) = \exp \left\{ \frac{1}{2\pi i} \int_C \frac{\ln [t^{-\nu} G(t)]}{t - z} dt \right\}.$$

It also follows from (13) that the functions $X^\pm(z)$ are continuously extendable to C , and $X^\pm(t) \in H(\alpha)$. Therefore $X^\pm(z) \in I(C, H(\alpha), 1)$. Rewrite (8) in the form

$$\Phi^+(t)[X^+(t)]^{-1} = t^\nu \Phi^-(t)[X^-(t)]^{-1} + g(t)[X^+(t)]^{-1}.$$

On the basis of Theorem 1,

$$g(t)[X^+(t)]^{-1} \in H_p^\alpha.$$

Putting

$$\Phi^+(z)[X^+(z)]^{-1} = \Phi_1^+(z), \quad z^\nu \Phi^-(z)[X^-(z)]^{-1} = \Phi_1^-(z),$$

we arrive at the jump problem with boundary condition

$$\Phi_1^+(t) - \Phi_1^-(t) = g(t)[X^+(t)]^{-1}, \quad (9)$$

in which the unknown function $\Phi_1^\pm(z)$ is analytic in D^+ and D^- , except at the infinitely distant point, where it has a pole of order ν , i.e., at infinity it has a principal part of the form

$$P_\nu(z) = a_0 + a_1 z + \dots + a_\nu z^\nu.$$

From the Sokhotski formulas it follows that the function

$$\Phi_1^\pm(z) = \frac{1}{2\pi i} \int_C \frac{g(t)}{X^+(t)} \frac{dt}{t - z} + P_\nu(z) \quad (10)$$

is a solution of problem (9). It belongs to the class $I(C, H_p^\alpha, P_\nu(z))$.

Let $\Phi_2^\pm(z)$ be another solution of the indicated problem in the same class of functions. Then, as follows from (4),

$$\Phi_2^\pm(z) = \frac{1}{2\pi i} \int_C \frac{\Phi_2^+(t) - \Phi_2^-(t)}{t - z} dt + P_\nu(z) = \frac{1}{2\pi i} \int_C \frac{g(t)}{X^+(t)} \frac{dt}{t - z} + P_\nu(z).$$

Consequently, the solution (10) of problem (9) in the class $I(C, H_p^\alpha, P_\varkappa(z))$ is unique (up to the coefficients of the polynomial $P_\varkappa(z)$).

From the solution found for the jump problem we obtain the solution of the Riemann problem (8) in the form

$$\Phi^+(z) = X^+(z)\Phi_1^+(z); \quad \Phi^-(z) = z^{-\varkappa}X^-(z)\Phi_1^-(z). \quad (11)$$

The complex constants $a_0, a_1, \dots, a_\varkappa$ in it are arbitrary.

Since $\Phi_1^\pm(z) \in I(C, H_p^\alpha, P_\varkappa(z))$ and $X^\pm(z) \in I(C, H(\alpha), 1)$, it follows, on the basis of Theorem 3, that $\Phi^\pm(z) \in I(C, H_p^\alpha, a_0)$.

- 2) Let $\varkappa < 0$. As above, we arrive at problem (9), where, however, $\Phi_1^-(z)$ has a zero of order $-\varkappa$ at infinity. The function

$$\Phi_1^\pm(z) = \frac{1}{2\pi i} \int_C \frac{g(t)}{X^+(t)} \frac{dt}{t-z}$$

will be the unique solution of problem (9) if the conditions

$$\int_C \frac{g(t)}{X^+(t)} t^{k-1} dt = 0, \quad k = 1, 2, \dots, -\varkappa - 1$$

are satisfied.

When these conditions are fulfilled, the solution of problem (8) is obtained from (11) with $P_\varkappa(z) \equiv 0$.

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REFERENCES

- ¹ Wu Xue-mo, *Hanshulyun yanjiu* (China), No. 1 (1958).
- ² N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.
- ³ F. D. Gakhov, *Boundary-Value Problems*, Moscow, 1963.
- ⁴ B. V. Khvedelidze, *Proceedings of the Tbilisi Mathematical Institute*, 23 (1956).
- ⁵ B. V. Khvedelidze, *Communications of the Academy of Sciences of the Georgian SSR*, 21, No. 2 (1958).
- ⁶ I. B. Simonenko, *Doklady AN*, 124, No. 2 (1959).
- ⁷ V. V. Ivanov, *Doklady AN*, 121, No. 5 (1958).
- ⁸ I. B. Simonenko, *Doklady AN*, 135, No. 3 (1960).
- ⁹ I. I. Danilyuk, *Siberian Mathematical Journal*, 1, No. 2 (1960).
- ¹⁰ T. G. Gegelia, *Communications of the Academy of Sciences of the Georgian SSR*, 15, No. 2 (1954).

¹¹ A. D. Alekseev, *Doklady AN*, 136, No. 3 (1960).

¹² A. D. Alekseev, *Proceedings of the Tbilisi Mathematical Institute*, 27 (1960).

¹³ T. G. Gegelia, *Communications of the Academy of Sciences of the Georgian SSR*, 13, No. 10 (1952).

Note: Figure translations are in progress. See original paper for figures.

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