

REPRESENTATION OF SPINORS IN $\backslash(n\backslash)$ -DIMENSIONAL SPACE BY SYSTEMS OF TENSORS

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Abstract

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MATHEMATICS

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REPRESENTATION OF SPINORS IN n - DIMENSIONAL SPACE BY SYSTEMS OF TENSORS

(Presented by Academician L. I. Sedov on 1 III 1966)

§ 1. **Basic definitions.** Let us first consider the even-dimensional complex Euclidean space R_n^+ , $n = 2\nu$, referred to an orthonormal basis e_i . Let $\gamma_1, \gamma_2, \dots, \gamma_{2\nu}$ be matrices of dimension 2^ν , which by definition satisfy the equation

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I, \quad (1)$$

where δ_{ij} is the Kronecker symbol, and I is the identity matrix of dimension 2^ν . Introduce the notation $\gamma_{i_1, i_2, \dots, i_k} = i^{k(k-1)/2} \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}$, $i_1 < i_2 < \dots < i_k$. The matrices $I, \gamma_i, \gamma_{i_1 \gamma_{i_2}}, \dots, \gamma_{i_1 \gamma_{i_2} \dots \gamma_{i_n}}$ are linearly independent and form a group of $2^{2\nu}$ elements. As is known, any two solutions $\gamma_i, \bar{\gamma}_i$ of equation (1) are connected by the equality $\bar{\gamma}_i = T \gamma_i T^{-1}$, $\det T \neq 0$, and the matrices γ_i may be chosen Hermitian and in such a way that $\gamma_1, \gamma_2, \dots, \gamma_\nu$ are symmetric, while $\gamma_{\nu+1}, \gamma_{\nu+2}, \dots, \gamma_{2\nu}$ are antisymmetric ⁽¹⁾.

Let $L = \|l_q^p\|$ be an orthogonal transformation of the space $R_{2\nu}^+$. The set of unimodular matrices S , defined by the equation

$$\gamma_p = l_p^q \gamma_q S^{-1}, \quad (2)$$

forms a group realizing a representation of the group L , called the **spin representation**.

An object $\psi = \{\psi^i\}$ with components ψ^i , defined up to sign and transforming according to the representation S , is called a **spinor of first rank** in the space $R_{2\nu}^+$.

If γ_i is a solution of (1), then, obviously, γ_i^T (γ_i^T are the transposed γ_i) is also a solution of (1); therefore there exists a matrix C such that $\gamma_i^T = C \gamma_i C^{-1}$, $\det C = 1$.

If ν is odd, then it is easy to see that $C = \gamma_\nu \gamma_{\nu-1} \cdots \gamma_1$; if ν is even, then $C = \gamma_{2\nu} \cdots \gamma_{\nu+1}$ in the case when $\gamma_1, \gamma_2, \dots, \gamma_\nu$ are symmetric and $\gamma_{\nu+1}, \gamma_{\nu+2}, \dots, \gamma_{2\nu}$ are antisymmetric. Hence it follows that $C^\tau = (-1)^{\nu(\nu-1)/2} C$. If ψ_i are the covariant components of the spinor ψ , then, by definition, $\psi_i = e_{ij} \psi^j$, where $E = \|e_{ij}\| = i^{n(n-1)/2} \gamma_n \gamma_{n-1} \cdots \gamma_1 C$.

In the space R_n^+ , select a pseudo-Euclidean space $R_n^{(s)}$ of index s , fixing the basis $ie_1 \dots ie_s e_{s+1} \dots e_n$. Introduce the Hermitian matrix Π , defined by the equations

$$\Pi \gamma_i \Pi^{-1} = \pm \gamma_i, \quad \Pi \Pi^* = (-1)^{(\nu-s)(\nu-s+1)/2} I; \quad (3)$$

here the minus sign is for $i = 1, \dots, s$, and the plus sign for $i = s+1, \dots, 2\nu$. A dot over a letter denotes complex conjugation. The spinor $\bar{\psi} = \Pi \psi$ is called **conjugate with respect to ψ** .

Let us now consider odd-dimensional spaces R_n^+ , $n = 2\nu+1$. We denote $\gamma_{2\nu+1} = i^\nu \gamma_{1,2,\dots,2\nu}$. The matrices with an even number of indices $I, \gamma_{i_1 i_2}, \dots, \gamma_{i_1 i_2 \dots i_{2\nu}}$ ($i_p = 1, 2, \dots, 2\nu+1$) are linearly independent. The spin representation of the proper orthogonal group L of transformations of the space $R_{2\nu+1}$ is given by the group of matrices S , defined from equation (2), in which the indices p, q take values from 1 to $2\nu+1$.

The covariant components of the spinor ψ_i are determined by the matrix E :

$$(-1)^\nu \gamma_i^T = E \gamma_i E^{-1}.$$

The conjugate spinor $\bar{\psi}$ in the space $R_{2\nu+1}^{(s)}$ is determined by the matrix Π

$$\pm \gamma_i = (-1)^{\nu-s} \Pi \gamma_i \Pi^{-1},$$

where the minus sign is for $i = 1, 2, \dots, s$.

§ 2. Representation of spinors by a system of complex tensors

Consider the complex matrix $\Psi = \{\psi^{ij}\}$ of dimension r . If $\psi^{ij} = \psi^i \psi^j$, then the ψ^{ij} satisfy the equalities

$$\psi^{ij} \psi^{kl} = \psi^{ik} \psi^{jl} = \psi^{il} \psi^{kj}, \quad \psi^{ij} = \psi^{ji}, \quad (4)$$

among which the independent ones are $r(r+1)/2 - r$ equations $\psi^{\nu\nu} \psi^{ij} = \psi^{\nu i} \psi^{\nu j}$ ($i, j \neq \nu$, $\psi^{\nu\nu} \neq 0$) and $\frac{1}{2}r(r-1)$ equations $\psi^{ij} = \psi^{ji}$. In all there are $r^2 - r$ independent equations. Conversely, it follows from (4) that there exists a system of r components ψ^k , determined up to sign, such that $\psi^{ij} = \psi^i \psi^j$. Indeed, if

$\psi^{\nu\nu} = 0$ for all ν , then from (4) it follows that $\psi^{ij} = 0$ for all values of the indices i, j . If $\psi^{\nu\nu} \neq 0$, then set

$$\psi^k = \psi^{\nu k} / \pm \sqrt{\psi^{\nu\nu}}. \quad (5)$$

In view of (4), this definition of ψ^k does not depend on the value of the index ν .

Suppose now that the ψ^{ij} are the components of an object Ψ transforming according to the representation $S \times S$, where S is any representation of some group, and let the equations (4) be invariant with respect to the group $S \times S$. Then the components ψ^k , defined according to (5), transform according to the representation S . Indeed, it follows from (5) that the transformation of ψ^k is determined by the transformation of ψ^{ij} in a unique way. Obviously, the invariance of the identities (4) is preserved if the ψ^k transform according to the representation S ; consequently, by uniqueness, the ψ^k can transform only according to the representation S . Thus, the object ψ^{ij} satisfying the identities (4) is equivalent to the object ψ^k .

Let S be the spinor representation of the 2ν -dimensional orthogonal group. It is known that

$$S \times S \sim \sum_{k=0}^{2\nu} D^k,$$

where D^k is the representation according to which the tensor of rank k , antisymmetric in all indices, transforms. Hence, in this case the object ψ^{ij} is equivalent to the tensor aggregate

$$\Lambda = \{c_0, c_i, c_{i_1 i_2}, \dots, c_{i_1 i_2 \dots i_{2\nu}}\}.$$

If S is the spinor representation of the $(2\nu + 1)$ -dimensional group, then

$$S \times S \sim \sum_{k=0}^{\nu} D^{2k};$$

therefore, the object ψ^{ij} is equivalent to a tensor aggregate consisting of tensors of even ranks.

The components of antisymmetric tensors $C_{i_1 i_2 \dots i_k}$ may be defined in the following way:

$$c_{i_1 i_2 \dots i_k} = (A_{i_1 i_2 \dots i_k})_{\alpha\beta} \psi^{\alpha\beta}, \quad A_{i_1 i_2 \dots i_k} = E \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}. \quad (6)$$

Since $\det E \neq 0$ and the $\gamma_{i_1 i_2 \dots i_k}$ are linearly independent, the $A_{i_1 i_2 \dots i_k}$ are also linearly independent, and, consequently, the aggregates Λ are indeed equivalent to the objects ψ^{ij} .

Using the symmetry properties of C , one can show that the matrices $A_{i_1 i_2 \dots i_k}$ have the following symmetry properties:

$$(A_{i_1 i_2 \dots i_k})^T = (-1)^{(\nu(\nu+1)+k(k+1))/2} A_{i_1 i_2 \dots i_k}.$$

Therefore, if ψ^{ij} satisfies the identities (4), then the part of the tensors $c_{i_1 i_2 \dots i_k}$ for which $[\nu(\nu+1)+k(k+1)]/2$ is odd vanishes. If ψ^{ij} satisfy the identities (4), then the tensors $c_{i_1 i_2 \dots i_k}$ satisfy $\frac{1}{2}2^\nu(2^\nu-1)$ independent bilinear identities, all of which contain—

be obtained in generalizing the Pauli identity for the case of an n -dimensional space [2]

$$2^\nu(\psi^+ \theta \psi)(\psi^+ \theta' \psi) = \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \dots < i_k}^{2\nu} (\psi^+ \gamma_{i_1 i_2 \dots i_k} \psi)(\psi^+ \theta' \gamma_{i_1 i_2 \dots i_k} \theta \psi) + (\psi^+ \psi)(\psi^+ \theta' \theta \psi), \quad n = 2\nu. \quad (7)$$

$$2^\nu(\psi^+ \varepsilon \psi)(\psi^+ \theta' \psi) = \sum_{k=1}^{2\nu} \sum_{i_1 < i_2 < \dots < i_k}^{2\nu+1} (\psi^+ \gamma_{i_1 i_2 \dots i_{2k}} \psi)(\psi^+ \theta' \gamma_{i_1 i_2 \dots i_{2k}} \theta \psi) + (\psi^+ \psi)(\psi^+ \theta' \theta \psi), \quad n = 2\nu + 1,$$

where θ' , θ are arbitrary matrices of dimension 2^ν ; ψ^+ , ψ are the covariant and contravariant components of a spinor.

Thus, a spinor ψ^k in the space R_n^+ , $n = 2\nu, 2\nu + 1$, is equivalent to a tensor aggregate Λ consisting of complex antisymmetric tensors satisfying the $\frac{1}{2}2^\nu(2^\nu-1)$ bilinear identities (7). In view of this, any spinor equation can be written in an equivalent manner as an equation in the components of the tensors $c_{i_1 i_2 \dots i_k}$.

We note that formula (5) determines the components ψ^k in any coordinate systems, but the transformation of the components ψ^k to curvilinear coordinates turns out to be nonlinear with respect to ψ^k .

§ 3. Representation of spinors by a system of real tensors. Consider an r -dimensional complex matrix $\psi^{\dot{p}q}$. Let $\psi^{\dot{p}q} = \psi^{\dot{p}} \psi^q$. Then $\psi^{\dot{p}q}$ satisfy the identities

$$\psi^{\dot{p}q} = (\psi^{\dot{q}p})^*, \quad (8)$$

among which the independent ones are $(r-1)^2$ real equations $\psi^{\dot{\nu}\nu} \psi^{\dot{p}q} = \psi^{\dot{\nu}q} \psi^{\dot{p}\nu}$ ($p, q \neq \nu$, $\psi^{\dot{\nu}\nu} \neq 0$) and r^2 real equations $\psi^{\dot{p}q} = (\psi^{\dot{q}p})^*$. Obviously, if the components ψ^k determine the matrix $\psi^{\dot{p}q}$, then the components $\psi^k e^{i\varphi}$, and only they, determine the same matrix $\psi^{\dot{p}q}$.

Conversely, it follows from (8) that there exists a system of components ψ^k , determined up to a phase $e^{i\varphi}$, such that $\psi^{\dot{p}q} = \psi^{\dot{p}}\psi^q$. Indeed, if $\psi^{\dot{\nu}\nu} = 0$ for all ν , then from (8) it follows that $\psi^{\dot{p}q} = 0$ for all p, q . In this case we put $\psi^k = 0$. If $\psi^{\dot{\nu}\nu} \neq 0$, then set

$$\psi^k = \frac{\psi^{\dot{\nu}k}}{\pm\sqrt{\psi^{\dot{\nu}\nu}}} e^{i\varphi}, \quad \varphi \text{ is an arbitrary real number.} \quad (9)$$

In virtue of (8), such a definition of the set ψ^k does not depend on the value of ν . It can be shown that only the components ψ^k determined by (9) satisfy the equation $\psi^{\dot{p}q} = \psi^{\dot{p}}\psi^q$.

Let $\psi^{\dot{p}q}$ be the components of an object transforming according to the representation $S \times S$, where S is any representation of some group, and let the equalities (8) be invariant with respect to the group $S \times S$. Then, obviously, one can specify such a transformation law φ that the components ψ^k transform according to the representation S .

Thus, specifying an object $\psi^{\dot{p}q}$ satisfying the identities (8), and the argument φ of one of the components ψ^k , completely determines the object ψ^k . Let S be the spinor representation of the 2ν -dimensional orthogonal group of transformations of the space $R_{2\nu}^{(s)}$. It is known that

$$S \times S \sim \sum_{k=0}^{2\nu} D^k.$$

Hence the object $\psi^{\dot{p}q}$ is equivalent to the tensor aggregate $\Omega = \{\Omega_0 \Omega_i \dots \Omega_{i_1 i_2 \dots i_2}\}$, consisting of antisymmetric tensors.

If S is a spinor representation in the space $R_{2\nu+1}^{(s)}$, then $S^* \times S \sim \sum_{k=0}^{\nu} D^{2k}$, and in this case the object ψ^{pq} is equivalent to the aggregate $\Omega = \{\Omega_0 \Omega_{i_1 i_2} \dots \Omega_{i_1 i_2 \dots i_{2\nu}}\}$, consisting of antisymmetric tensors of even rank.

The components of these tensors may be defined as follows:

$$\Omega_{i_1 i_2 \dots i_k} = (D_{i_1 i_2 \dots i_k})_{\alpha} \psi^{\alpha\beta}, \quad D_{i_1 i_2 \dots i_k} = E \Pi \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k} i^{k(k+1)/2}.$$

Using (3), one can show that the matrices $D_{i_1 i_2 \dots i_k}$ are Hermitian.

Therefore, if ψ^{pq} satisfy the identity $\psi^{pq} = (\psi^{qp})^*$, then the components $\Omega_{i_1 i_2 \dots i_k}$ are real. If ψ^{pq} satisfy the identities (8), then the components of the tensors $\Omega_{i_1 i_2 \dots i_k}$ satisfy $(2\nu-1)^2$ bilinear identities, each of which is contained in identity (7).

Thus, specifying the aggregate Ω and the argument φ of one of the components ψ^k completely determines the spinor. Hence, the spinor equations can be written

in an equivalent form in the components of the aggregate Ω and φ . Eliminating the argument φ from such equations, one can then obtain a closed system of equations in the components of the aggregate Ω .

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Note: Figure translations are in progress. See original paper for figures.

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