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V. P. Kompaniets, A. V. Chernavskii

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Abstract

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MATHEMATICS

V. P. KOMPANIETS, A. V. CHERNAVSKII

EQUIVALENCE OF TWO CLASSES OF MAPPINGS OF THE SPHERE

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In this note we consider a class of mappings of the n -dimensional sphere onto itself, interest in which arose in connection with the hypothesis that mappings of this class coincide with mappings that can be approximated by homeomorphisms (see ⁽¹⁾, p. 169). It is known that for $n = 1$ and 2 the class of mappings approximable by homeomorphisms coincides with the monotone mappings of the sphere onto the sphere ⁽²⁾. Already for $n = 3$ this is not so, as is shown by Bing's example ⁽³⁾, p. 7) and by a more complicated construction of L. V. Keldysh (see ⁽¹⁾, p. 169). On the other hand, in ⁽⁴⁾ it is shown that mappings that can be approximated by homeomorphisms have the following property:

(K) The preimage of every cellular set is a cellular set.

A subset of S^n is called cellular ⁽⁵⁾ if it is the intersection of a sequence of n -dimensional cells, in which each following one lies inside the preceding one. We shall call a mapping of S^n into itself cellular if it has property (K).

A mapping of S^n was called pointlike by Bing if it has the following property:

(T) The preimage of every point is a cellular set.

It is clear that property (K) implies property (T). We prove the converse:

Theorem. *For $n \neq 4$, a pointlike mapping of the sphere onto itself is cellular.*

For $n \leq 2$ every acyclic continuum in the sphere is cellular and, under monotone mappings of the sphere onto itself, the preimages of points are acyclic; therefore both classes of mappings coincide with the monotone mappings of the sphere onto itself and hence coincide with each other. For $n = 3$ and $n > 4$ we rely on the characterization of Euclidean space given by Edwards ⁽⁸⁾ and Stallings ⁽⁶⁾, respectively. In addition, in both cases we rely on Lemma 1 proved below, which is an application to the present situation of the technique used by Smale in the proof of the "homotopy theorem of Vietoris" ⁽⁷⁾.

Lemma 0. *Let K be a continuum lying in the sphere S^n . The following properties of K are equivalent:*

1. K is a cellular continuum.
2. $S^n \setminus K$ is homeomorphic to R^n .
3. The quotient space of the continuous decomposition of the sphere into K and the points of the complement of K is homeomorphic to S^n .

The equivalence of 1 and 3 is established in ⁽⁵⁾, and the equivalence of 2 and 3 is obvious. We shall need the equivalence of 1 and 2.

Proof of the theorem. Let a pointlike mapping of the sphere onto itself be given, and let a cellular continuum K in S^n be given. We note that, by property 2 of Lemma 0 and Alexander duality, cellular continua are acyclic, and therefore, by the Vietoris theorem, the homology groups of fS^n coincide with the homology groups of S^n . Hence, $fS^n = S^n$. We must prove that,

that $f^{-1}K$ is a cellular continuum. We shall again use the equivalence of properties 1 and 2 of Lemma 0 and show that $S^n \setminus K$ is homeomorphic to R^n . For this, by Stallings' criterion ⁽⁶⁾, it is enough to establish that all $\pi_i(S^n \setminus f^{-1}K) = 0$ and that $S^n \setminus f^{-1}K$ is unknotted at infinity, i.e., that for every compact set $C \subset S^n \setminus f^{-1}K$ there is a compact set $D \subset S^n \setminus f^{-1}K$, containing C , such that $S^n \setminus D \setminus f^{-1}K$ is connected and simply connected. The proofs of all these assertions are based on the application of the following lemma.

Lemma 1. Let $f : S^n \rightarrow S^n$ be a point mapping; P a finite polyhedron; Q a subpolyhedron, and let $\varphi : P \rightarrow S^n$, $\Psi : Q \rightarrow S^n$ be such that $\psi = f\Psi = \varphi|_Q$. Then for every $\varepsilon > 0$ there exists a mapping $\Phi : P \rightarrow S^n$ such that $\Phi|_Q = \Psi$ and $\rho(f\Phi, \varphi) < \varepsilon$, where ρ is the metric in the space of mappings of P into S^n .

We first show how the proof of the theorem is completed with the help of the lemma.

Let us show that $\pi_i(S^n \setminus f^{-1}K) = 0$. Let $\Psi : S^i \rightarrow S^n \setminus f^{-1}K$ be any continuous mapping of the boundary of an $(i+1)$ -cell B^{i+1} , and let $\psi = f\Psi$. Since K is a cellular continuum and $S^n \setminus K$ is homeomorphic to R^n , there exists an extension $\varphi : B^{i+1} \rightarrow S^n \setminus K$ of the mapping ψ . By the lemma, for every $\varepsilon > 0$ there exists an extension $\Phi : B^{i+1} \rightarrow S^n$ of the mapping Ψ such that $\rho(f\Phi, \varphi) < \varepsilon$. If $\varepsilon < d(\varphi(B^{i+1}), K)$, then $\Phi(B^{i+1})$ lies outside $f^{-1}K$, and since $\Psi : S^i \rightarrow S^n \setminus f^{-1}K$ was chosen arbitrarily, $\pi_i(S^n \setminus f^{-1}K) = 0$.

Let us show that $S^n \setminus f^{-1}K$ is unknotted at infinity. Let C be an arbitrary compact set in $S^n \setminus f^{-1}K$, and let V be such a cellular neighborhood of K that $f^{-1}V \subset S^n \setminus C$. Put $D = f^{-1}(S^n \setminus V)$ and show that $\pi_1(S^n \setminus f^{-1}K \setminus D) = 1$. As above, let $\Psi : S^1 \rightarrow S^n \setminus f^{-1}K \setminus D$ be a mapping of the boundary S^1 of the disk B^2 into $S^n \setminus f^{-1}K \setminus D$, and let $\psi = f\Psi$. It is clear that $\psi(S^1) \subset V \setminus K$. Since $n > 2$, K is cellular and V is a cellular neighborhood of K , there exists an extension $\varphi : B^2 \rightarrow V \setminus K$ of the mapping ψ . By the lemma, for every $\varepsilon = 0$ there exists such an extension $\Phi : B^2 \rightarrow S^n$ of the mapping Ψ that $\rho(f\Phi, \varphi) < \varepsilon$. If $\varepsilon < d(\varphi(D^2), K \cup (S^n \setminus V))$, then $f\Phi(D^2) \subset S^n \setminus K$ and, consequently, $\Phi(D^2) \subset S^n \setminus f^{-1}K \setminus D$. Since Ψ was arbitrary, $S^n \setminus f^{-1}K \setminus D$ is simply connected. In exactly the same way it is shown that $S^n \setminus f^{-1}K \setminus D$

connected. Since C was an arbitrary compact set, $S^n \setminus f^{-1}K$ is unknotted at infinity. The theorem is proved.

Proof of Lemma 1. We shall use induction on the dimension of the polyhedron. As usual in such cases, it is enough to consider the situation when P is a k -dimensional cell and Q is its boundary. Put $P = B^k$, $Q = S^{k-1}$. If $k = 0$, then B^k is a point, and for $\Phi(B^k)$ one should take an arbitrary point in $f^{-1}(\varphi B^k)$. Then $f\Phi = \varphi$. Suppose that in dimensions less than k the lemma is true. Let $\varepsilon > 0$ be given. For each point $x \in S^n$ there is a neighborhood $U(x)$ such that $f^{-1}(\overline{Ux})$ lies in such a cellular neighborhood $V(x)$ that the diameter of $f(V(x))$ is less than $\varepsilon/3$. Choose from the sets $U(x)$ a finite covering $\{U_i\}$ and denote by δ the minimum of $\varepsilon/3$ and the Lebesgue number of this covering. Using the inductive assumption, we can construct an extension Ψ^{k-1} of the mapping Ψ to the $(k-1)$ -dimensional skeleton so fine a triangulation T of the cell B^k that the image of each simplex of T has diameter less than $\delta/3$. In doing so the extension Ψ^{k-1} may be chosen so that the requirement

$$\rho(f\Psi^{k-1}, \varphi|_{|T|^{k-1}}) < \delta/3,$$

is satisfied, where $|T|^{k-1}$ is the $(k-1)$ -skeleton of T . The mapping Ψ^{k-1} has the property that the diameter of the image of the boundary of each k -simplex of the triangulation T is less than δ , and therefore each such image of the boundary of a simplex lies in some neighborhood U_i . By assumption, $f^{-1}(U_i)$ lies in the cell V_i , the diameter of the image of which is less than $\varepsilon/3$. Let Δ^k be an arbitrary k -simplex of T . $\Psi^{k-1}(\dot{\Delta}^k)$ lies in some $f^{-1}(U_i)$ and, consequently, in some cell V_i . Hence one can construct an extension $\Phi|_{\Delta^k}$ of the mapping Ψ^{k-1} , for which the image $\Phi(\Delta^k)$ lies in V_i . Thus Φ can be constructed on the whole cell B^k . We shall show that $\rho(f\Phi, \varphi) < \varepsilon$.

Indeed, if the point $x \in \Delta^k$, where Δ^k is some closed simplex of the triangulation, then the diameters of $\varphi(\Delta^k)$ and $f\Phi(\Delta^k)$ are less than $\varepsilon/3$, and the distance between the images of the boundary of Δ^k under the mappings φ and $f\Phi|_{\Delta^k} = f\Psi^{k-1}$ is also less than $\varepsilon/3$. Therefore the distance between $\varphi(x)$ and $f\Phi(x)$ is less than $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. The lemma is proved.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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