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Abstract

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MATHEMATICS

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ON GAUSSIAN MEASURES EQUIVALENT TO GAUSSIAN MARKOV MEASURES

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The question of conditions for the equivalence or orthogonality of Gaussian measures in the space of real functions has attracted the attention of many authors (see, for example, the survey articles (⁵, ¹⁵)). Especially much attention has been devoted to the special case when one of the measures is Wiener measure—this case was considered in (⁷, ¹³, ¹⁴) and in a number of other works. In particular, in (¹³) conditions were obtained for the equivalence of a given Gaussian measure to Wiener measure for a broad class of Gaussian measures. However, the conditions obtained there are imposed not on the correlation function $R(s, t)$, which is the most natural characteristic of a Gaussian measure, but on a certain function defined from $R(s, t)$ by means of a nonlinear integral equation.

In the present note we indicate conditions which the function $R(s, t)$ must satisfy in order that the corresponding Gaussian measure be equivalent to Wiener measure. Starting from this, conditions of equivalence or orthogonality are established for two Gaussian measures, one of which corresponds to an arbitrary Gaussian Markov process (the case of two measures, one of which corresponds to a stationary Gaussian Markov process and the other to an arbitrary Gaussian stationary process, was studied earlier in (⁴)).

1. Let (Ω, B) be a measurable space induced by a family of random variables ξ_t , $t \in [a, b]$. Denote by P_1 and P_2 probability measures on (Ω, B) with respect to which ξ_t is a Gaussian process with $M_1 \xi_t \equiv 0$, $M_1 \xi_s \cdot \xi_t = R_1(s, t)$, and $M_2 \xi_t \equiv 0$, $M_2 \xi_s \cdot \xi_t = R_2(s, t)$, where M_1 and M_2 are the expectations corresponding to the measures P_1 and P_2 , respectively. Let us introduce the spaces $H_1(\xi)$ and $H_2(\xi)$ —the linear closures of ξ_t , $t \in [a, b]$, with respect to the norms corresponding to the scalar products $(\gamma, \eta)_1 = M_1 \gamma \cdot \eta$ and $(\gamma, \eta)_2 = M_2 \gamma \cdot \eta$. Suppose that $(\gamma, \eta)_2 = (S\gamma, \eta)_1$, where S is a symmetric operator in $H_1(\xi)$. In this case, as shown in (³, ⁸, ¹²), the measures P_2 and P_1 will be equivalent (this will be denoted by $P_2 \sim P_1$) if and only if: 1) the operator S is bounded and has a bounded inverse, and 2) the operator $I - S$ is a Hilbert-Schmidt operator.

The method of proof of Theorem 1 below consists in constructing Hilbert spaces

$H_1(R)$ and $H_2(R)$, isomorphic to $H_1(\xi)$ and $H_2(\xi)$, with reproducing kernels $R_1(s, t)$ and $R_2(s, t)$ (see ^(6, 9, 10)). To the operator S in $H_1(\xi)$ there naturally corresponds an operator $\bar{S}$ in $H_1(R)$; the latter is defined explicitly in terms of the functions $R_1(s, t)$ and $R_2(s, t)$. Starting from this, the conditions for equivalence of the measures P_1 and P_2 can be expressed directly through the correlation functions. We shall consider only the case when, with respect to the measure P_1 , the process ξ_t is a mean-square continuous Gaussian Markov process.

Let, first of all, with respect to the measure P_1 , the process $\xi_t \equiv \xi(t)$ be a Wiener process, i.e. $R_1(s, t) = \min(s, t)$, $s, t \in [0, T] \subseteq [0, \infty]$.

Theorem 1. $P_2 \sim P_1$ if and only if:

- 1) $R_2(0, t) \equiv 0$.
- 2) $R_2(s, t)$ is an absolutely continuous function in each argument on any finite interval contained in $[0, T]$, and moreover

$$\frac{\partial}{\partial t} R_2(s, t) \in \mathcal{L}^2(0, T).$$

The unique solution in $\mathcal{L}^2(0, T)$ of the equation

$$\int_0^T f(t) \frac{\partial}{\partial t} R_2(s, t) dt = 0$$

is $f(t) \equiv 0$.

- 3) $\frac{\partial}{\partial t} R_2(s, t)$ is an absolutely continuous function of s for $0 \leq s < t$, $t < s \leq T$ for almost all $t \in [0, T]$.

4)

$$\frac{\partial}{\partial t} R_2(s, t) \Big|_{s=t+0} - \frac{\partial}{\partial t} R_2(s, t) \Big|_{s=t-0} = 1,$$

where the partial derivatives are understood as one-sided derivatives.

5)

$$\iint_{0 \leq s \leq t \leq T} \left[\frac{\partial^2}{\partial s \partial t} R_2(s, t) \right]^2 ds dt < \infty.$$

This theorem contains, as special cases, the main results of the papers ^(13,14). The proof of the theorem is based on the following lemmas.

Lemma 1. $\gamma \in H_1(\xi)$ if and only if

$$\gamma = \int_0^T \varphi(t) d\xi(t)^*,$$

where $\varphi(t) \in \mathcal{L}^2$, $\xi(t)$ is a Wiener process. If $P_2 \sim P_1$, then there exists

$$\int_0^T \varphi(t) d\xi(t),$$

where $\varphi(t) \in \mathcal{L}^2$, $\xi(t)$ is a process defined by the correlation function $R_2(s, t)$, and hence

$$\int_0^T \varphi(t) d\xi(t) = \gamma \in H_2(\xi).$$

In this case $H_2(\xi)$ consists only of random variables γ representable in this form.

The assertion concerning $H_1(\xi)$ is a reformulation of a well-known property of the stochastic integral (see, for example, ⁽¹⁾, p. 383). The assertion concerning $H_2(\xi)$ follows from the fact that, if $P_2 \sim P_1$, then between $H_1(\xi)$ and $H_2(\xi)$ there exists a one-to-one continuous correspondence. Denote by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ the scalar products in $H_1(R)$ and $H_2(R)$.

Lemma 2. The space $H_1(R)$ consists of absolutely continuous functions $\varphi(t)$ such that $\varphi(0) = 0$, $\varphi'(t) \in \mathcal{L}^2$; moreover

$$\langle g(t), h(t) \rangle_1 = \int_0^T g'(t) h'(t) dt.$$

The isomorphism between $H_1(R)$ and $H_1(\xi)$ is given by the equality

$$H_1(R) \ni \varphi(t) \leftrightarrow \gamma = \int_0^T \varphi'(t) d\xi(t) \in H_1(\xi).$$

If $P_2 \sim P_1$, then $H_2(R)$ consists of the same functions as $H_1(R)$.

The proof is based on the fact that the correspondence between $\gamma \in H_1(\xi)$ and $\varphi(t)$, established by the equality

$$M_1 \gamma \cdot \xi(t) = \varphi(t),$$

is an isomorphism between $H_1(\xi)$ and $H_1(R)$.

Lemma 3. If $P_2 \sim P_1$, then the function $R_2(s, t)$ is differentiable in each argument, and

$$\bar{S}g(s) = \int_0^T g'(t) \frac{\partial}{\partial t} R_2(s, t) dt,$$

* The integral here and below in analogous cases is understood as an integral in the mean-square sense.

$$(I - \bar{S})g(s) = \int_0^T g'(t) \frac{\partial}{\partial t} [\min(s, t) - R_2(s, t)] dt.$$

The proof easily follows from the equalities $M_1\gamma \cdot \xi(t) = g(t)$ and $M_2\gamma \cdot \xi(t) = \bar{S}g(t)$.

Lemma 4. Let $A(s, \cdot) \in H_1(R)$. In order that in the space $H_1(R)$ an operator A of the form

$$Ag(s) = \int_0^T g'(t) \frac{\partial}{\partial t} A(s, t) dt$$

be a Hilbert-Schmidt operator, it is necessary and sufficient that $\frac{\partial}{\partial t} A(s, t)$ be absolutely continuous in s for almost all t , and that the inequality

$$\int_0^T \int_0^T \left[\frac{\partial^2}{\partial s \partial t} A(s, t) \right]^2 ds dt < \infty$$

hold.

The proof of the lemma follows from the fact that the correspondence

$$H_1(R) \ni g(t) \leftrightarrow g'(t) \in \mathcal{L}^2$$

is an isomorphism under which to the operator A in $H_1(R)$ there corresponds the operator

$$\bar{A}f(s) = \frac{d}{ds} \int_0^T f(t) \frac{\partial}{\partial t} A(s, t) dt \quad \text{in } \mathcal{L}^2.$$

Applying Lemma 4 to the operator $I - \bar{S}$, defined in Lemma 3, we obtain the assertion of Theorem 1.

2. Let now the process $\xi(t)$, $t \in [a, b]$, be a Gaussian Markov process with respect to the measure P_1 . Such a process is nondegenerate (i.e., satisfies the condition $D_1\xi(t) \neq 0$) on an open set composed of intervals (α_k, β_k) , $k = 1, 2, \dots$, on each of which the correlation function can be represented in the form

$$R_1(s, t) = \psi_k(s) \cdot \varphi_k(t), \quad \alpha_k \leq s \leq t \leq \beta_k,$$

where $\psi_k(t)$ and $\varphi_k(t)$ are continuous functions such that $\psi_k(t)/\varphi_k(t)$ does not decrease for $t \in [\alpha_k, \beta_k]$. For convenience we consider only the case of one interval of nondegeneracy. Let

$$g(\psi(t)/\varphi(t)) = t,$$

$$\mu^*(u, v) = R_2(g(u), g(v))/\varphi(g(u)) \cdot \varphi(g(v)),$$

$$\frac{\partial}{\partial u} \mu^*(u, v) \Big|_{u=\psi(s)/\varphi(s), v=\psi(t)/\varphi(t)} = \mu_u(s, t)$$

and the notation $\mu_v(s, t)$ and $\mu_{uv}(s, t)$ have an analogous meaning. Further, $H_1(R)$ denotes the space reproduced by the correlation function

$$R_1(s, t) = \psi(s)\varphi(t), \quad a \leq s \leq t \leq b.$$

Theorem 2. $P_2 \sim P_1$ if and only if:

1)

$$\frac{R_2(a, t)}{\varphi(a)\varphi(t)} = 0$$

* if and only if

$$\frac{\psi(a)}{\varphi(a)} = 0.$$

2) $R_2(s, \cdot) \in H_1(R)$. This means, first, that $\mu^*(u, v)$ is an absolutely continuous function in each argument on any finite interval contained in the set of values of $\psi(t)/\varphi(t)$, $t \in [a, b]$; second, that

$$\int_a^b [\mu_v(s, t)]^2 d\frac{\psi(t)}{\varphi(t)} + \frac{R_2^2(s, a)}{\varphi^2(s)\psi(a)\varphi(a)} < \infty.$$

The equation

$$\int_a^b F(t)\mu_v(s, t) d\frac{\psi(t)}{\varphi(t)} = \lambda \frac{R_2(a, s)}{\varphi(a)\varphi(s)}$$

* $\Phi(a, t)$ denotes $\lim_{s \rightarrow a+0} \Phi(s, t)$.

The notation $\Phi(s, a)$ and $f(a)$ has an analogous meaning.

(where, for $R_2(a, s)/\varphi(a)\varphi(s) = 0$, we put $\lambda = 0$) has in $L_2(\psi/\varphi)$ the unique solution $\lambda = 0$, $F(t) \equiv 0$.

3) For almost all v , $\mu_v^*(u, v)$ is an absolutely continuous function of u for $\psi(a)/\varphi(a) \leq u < v$ and for $v < u \leq \psi(b)/\varphi(b)$.

4) $\mu_v(s, t)|_{s=t+0} - \mu_v(s, t)|_{s=t-0} = 1$.

5)

$$\iint_{a \leq s \leq t \leq b} [\mu_{uv}(s, t)]^2 d\frac{\psi(s)}{\varphi(s)} d\frac{\psi(t)}{\varphi(t)} < \infty.$$

The proof of the theorem is based on the fact that the process

$$\omega(u) = \xi[g(u)]/\varphi[g(u)]$$

is a Wiener process on the interval

$$[\psi(a)/\varphi(a), \psi(b)/\varphi(b)]$$

under the condition that the function $\psi(t)/\varphi(t)$, $t \in [a, b]$, is strictly increasing. Using the method justified in ⁽²⁾, we arrive at the case considered in Theorem 1. We note that Theorem 2 contains, in particular, the result of ⁽⁴⁾, pertaining to the case of spectral density

$$f(\lambda) = c(\lambda^2 + a^2)^{-1}.$$

3. In the case of Gaussian measures P_1 and P_2 , the computation of the likelihood ratio (the Radon-Nikodym derivative of the measures P_1 and P_2), which is important for statistical applications, is simplified if the measures P_1 and P_2 are not only equivalent, but also strongly equivalent in the sense of Hájek ⁽⁹⁾ (denoted $P_1 \approx P_2$).

Let, with respect to the measure P_1 , the process $\xi(t)$ be Wiener, and with respect to the measure P_2 Gaussian with correlation function

$$R_2(s, t) = \min(s, t) + R(s, t),$$

where $R(s, t)$ is a positive definite kernel (such a situation often occurs in statistical applications of the theory of stochastic processes). In this case the following holds.

Theorem 3. $P_2 \approx P_1$ if and only if conditions 1)–4) of Theorem 1 and the condition

5') For almost all $t \in [0, T]$ there exists

$$\left. \frac{\partial^2}{\partial s \partial t} R(s, t) \right|_{s=t}$$

and

$$\int_0^T \left| \left. \frac{\partial^2}{\partial s \partial t} R(s, t) \right|_{s=t} \right| dt < \infty.$$

Let, with respect to the measure P_1 , the process $\xi(t)$, $t \in [a, b]$, be a Gaussian Markov process with correlation function

$$R_1(s, t) = \psi(s)\varphi(t), \quad a \leq s \leq t \leq b,$$

where $\psi(s) \neq 0$, $\varphi(s) \neq 0$ for $s \in (a, b)$. With respect to the measure P_2 , the process $\xi(t)$ is a Gaussian process with correlation function

$$R_2(s, t) = R_1(s, t) + R(s, t),$$

where $R(s, t)$ is a positive definite kernel. From $R(s, t)$ form the function $v^*(u, v)$ (see Sec. 2). In this case the following is true.

Theorem 4. $P_2 \approx P_1$ if and only if conditions 1)–4) of Theorem 2 and the following condition are satisfied:

5'') For almost all $v \in [\psi(a)/\varphi(a), \psi(b)/\varphi(b)]$ there exists

$$v_{uv}^*(u, v)|_{u=v}$$

and

$$\int_a^b |v_{uv}(t, t)| d \frac{\psi(t)}{\varphi(t)} < \infty.$$

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