

HOMOLOGIES OF SPACES OF SMOOTH EMBEDDINGS

A. S. Schwarz

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Abstract

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MATHEMATICS

A. S. SHVARTS

HOMOLOGIES OF SPACES OF SMOOTH EMBEDDINGS

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In the present note we outline proofs of some theorems that make it possible to obtain information on the homologies of the space of smooth embeddings of a smooth manifold V^k into a smooth manifold W^n in the case when $n > 2k + 1$.^{*} As an application of these theorems, the homology groups of the space of embeddings of the circle S^1 in the sphere S^n are computed in dimensions $\leq 3n - 10$.

By V^k and W^n we shall henceforth denote smooth manifolds of dimensions k and n , respectively; the manifold V^k will be assumed compact. By $Pl(V^k, W^n)$ we denote the space of smooth embeddings of V^k in W^n , endowed with the C^r -topology,^{**} and by $Pl_0(V^k, W^n)$, or simply Pl_0 , the space of based embeddings, i.e. the subspace of the space Pl consisting of embeddings that carry a marked point of V^k to a marked point of W^n and have at this point the prescribed differential.

The manifolds V^k and W^n with small open balls, correctly situated in them, removed will be denoted by V_1^k and W_1^n ; the space of immersions of V_1^k into W_1^n that carry the boundary of the manifold V_1^k , by means of a standard diffeomorphism, into a certain sphere S^{k-1} lying on the boundary of the manifold W_1^n , in such a way that the image of V_1^k does not meet the boundary of W_1^n , will be denoted by $Im(V^k, W^n)$, or simply Im . (It is easy to see that Pl_0 is homotopy equivalent to the set $A_0 \subset Im$, consisting of embeddings (see (1)).)

We shall also use the notation $Im_1^r = Im_1^r(V^k, W^n)$ for the subset of $Im(V^k, W^n)$ consisting of immersions with one multiple point of multiplicity r ; A_1 for the subset of Im_1^2 consisting of immersions for which the tangent planes to the image of V_1^k at the double point are in general position.

Proposition 1. *If $n \geq 2k + 1$, then in dimensions $i \leq 2n - 4k - 2$ there is an exact sequence*

$$H_{i+1-n+2k}(A_1; Z_2) \rightarrow H_i(Pl_0; Z_2) \rightarrow H_i(Im; Z_2) \rightarrow H_{i-n+2k}(A_1; Z_2).$$

For the proof of this assertion we first note that the spaces $A_0 \cup A_1$ and A_1 may be regarded as infinite-dimensional smooth manifolds, with A_1 being a submanifold of codimension $n - 2k$ in the manifold $A_0 \cup A_1$ and closed in $A_0 \cup A_1$. From this fact follows the relation

$$H_i(A_0 \cup A_1, A_0; Z_2) = H_{i-n+2k}(A_1; Z_2)$$

(see, for example, ⁽²⁾). Further, we note that

$$H_i(A_0 \cup A_1; Z_2) = H_i(Im; Z_2)$$

for $i \leq 2n - 4k - 2$ (this follows from the fact that the manifold $A_0 \cup A_1$ is obtained from the manifold Im by deleting a submanifold having, in a certain sense, codimension $2n - 4k$; the careful proof is easiest to carry out by passing to a finite-dimensional approximation). The required

* By the word smooth we mean differentiable a sufficiently large number of times.

** The number r is assumed sufficiently large; the homotopy properties of the spaces $Pl(V^k, W^n)$ and $Pl_0(V^k, W^n)$ do not depend on the choice of the number $r \geq 1$.

the assertion is obtained if one considers the exact sequence of the pair $(A_0 \cup A_1, A_0)$ and uses the relations indicated above.

Remark 1. It is convenient to study the homology of the space A_1 by means of the fibration obtained by assigning to each immersion f from A_1 the pair of points of the manifold V_1^k that are “glued together” under the immersion f . The base of this fibration is the space V_2 of unordered pairs of distinct points of the manifold V_1^k ; its fiber, in turn, may be regarded as the space of a fibration whose base is the space of $2k$ -frames tangent to the manifold W_1^n , while the fiber in dimensions $\leq n - 2k - 2$ (i.e., in almost all dimensions of interest to us) has the same homology groups as the space of a triad of based immersions of the manifold V^k into the manifold W^n (i.e., immersions taking three marked points of V^k to three marked points of the manifold W^n and having prescribed differentials at these points).

Remark 2. The exact sequence of Proposition 1 also holds for any coefficient group G , if one regards the homology groups of the space A_1 as being taken with respect to the corresponding, suitably chosen local coefficient system $\{G\}$. Using the considerations of Remark 1, one can compute the fundamental group of the space A_1 ; in particular, if $n > 2k + 1$ and W^n is a simply connected manifold, then $\pi_1(A_1) = \pi_1(V_2)$.

Consider in the group $\pi_1(V_2)$ the image R of the fundamental group of the space $\tilde{V}_2$ of ordered pairs of distinct points of the manifold V_1^k under the natural map $\tilde{V}_2 \rightarrow V_2$. The action of the group $\pi_1(A_1)$ on the group G , which produces

the local coefficient system we need, is described as follows: to the elements of the subgroup $R \subset \pi_1(V_2) = \pi_1(A_1)$ one assigns the identity transformation of the group G , and to the remaining elements the automorphism $\alpha(g) = -g$. In particular, if the manifold V^k is homeomorphic to the circle S^1 , then the local coefficient system G is trivial (this is easily seen directly).

Let us note that Proposition 1 immediately implies the main result of the note by A. M. Vinogradov (1): the space $Pl_0(S^1, S^n)$ is acyclic in dimensions $\leq 2n-7$. Indeed, in this case Im is homotopy equivalent to $\Omega(S^{n-1})$, i.e. $H_i(Im; \mathbb{Z}) = 0$ for $i \neq k(n-2)$, $H_{k(n-2)}(Im; \mathbb{Z}) = \mathbb{Z}$. It is also easy to verify that A_1 is acyclic in dimensions $< n-3$; therefore, in order to compute the groups $H_i(Pl_0; \mathbb{Z})$ for $i \leq 2n-7$, it is necessary only to study the homomorphism $H_{n-2}(Im; \mathbb{Z}) \rightarrow H_0(A_1; \mathbb{Z})$ occurring in the exact sequence of Proposition 1. By a direct geometric construction it is easily shown that this homomorphism is an isomorphism.

Proposition 1 can also be used to obtain the following more general result.

Proposition 2. *The space $Pl_0(S^k, S^n)$ of based embeddings of the sphere S^k in the sphere S^n is acyclic in dimensions $< 2n-4k-2$.*

However, a simpler proof of this assertion will be sketched below on the basis of somewhat different considerations.

One can construct a spectral sequence containing within it the exact sequence of Proposition 1 and making it possible to obtain information about the homology groups of the space Pl_0 also in dimensions higher than $2n-4k-2$. For simplicity, we shall formulate the corresponding assertion only for spaces of embeddings of the circle and for dimensions $\leq 3n-8$.

Thus, let $V^k = S^1$; let A_0 and A_1 be the sets constructed above; let A_2 be the subset of $Im(S^1, W^n)$ consisting of immersions with two multiple points of multiplicity 2, at each of which the tangents do not coincide; let A_3 be the subset Im_1^2 , consisting of immersions having first-order contact at the multiple point; let A_4 be the subset Im_1^3 , characterizing-

...by the fact that, for the immersions entering it, the tangent lines at the multiple point are in general position.

Proposition 3. There exists a spectral sequence $\{E_r^{p,q}\}$, in which

$$E_1^{0,q} = H_q(Pl_0(S^1, W^n); \mathbb{Z}), \quad E_1^{1,q} = H_{q-n+3}(A_1; \mathbb{Z}),$$

$$E_1^{2,q} = H_{q-2n+6}(A_2; \mathbb{Z}), \quad E_1^{3,q} = H_{q-2n+6}(A_3 \cup A_4; \mathbb{Z}), \quad E_1^{p,q} = 0$$

for $p \geq 4$ and $p < 0$, and for $l \leq 3n-8$ the group

$$\sum_{p+q=l} E_\infty^{p,q}$$

is a group associated with the group $H_l(Im(S^1, W^n); \mathbb{Z})$ in some filtration.

The proof is based on the following considerations. Consider in the space $Im(S^1, W^n)$ the subspace

$$E = \bigcup_{0 \leq i \leq 4} A_i$$

and in this subspace the filtration $E_0 = A_0$, $E_1 = A_0 \cup A_1$, $E_2 = A_0 \cup A_1 \cup A_2$, $E_3 = E$. As is known, with the aid of such a filtration one can construct a spectral sequence for which

$$E_1^{p,q} = H_{p+q}(E_p, E_{p-1}; Z),$$

and

$$\sum_{p+q=l} E_{\infty}^{p,q}$$

is a group associated with the group $H_l(E; Z)$ endowed with a certain filtration. This is the spectral sequence we need, since

$$H_l(E; Z) = H_l(Im; Z)$$

for $l \leq 3n - 8$ (because $Im \setminus E$ has, in E , in a certain sense codimension $3n - 6$), and the homology groups $H_q(E_p, E_{p-1}; Z)$ are expressed in terms of the homology groups $H_s(E_p \setminus E_{p-1}; Z)$, in view of the fact that E_p may be regarded as an infinite-dimensional smooth manifold, and $E_p \setminus E_{p-1}$ as a closed submanifold of the manifold E_p . (The local system of coefficients on $E_p \setminus E_{p-1}$ turns out to be trivial.)

Proposition 4. The space $Pl_0(S^1, S^n)$ for $n \geq 4$ is acyclic in all dimensions $\leq 3n - 10$, with the exception of dimension $2n - 6$; the group

$$H_{2n-6}(Pl_0(S^1, S^n); Z)$$

is a free cyclic group.

For the proof, note that, in the dimensions of interest to us, in the spectral sequence of Proposition 3 among the groups $E_1^{p,q}$ with $p \geq 1$ there are only the following nonzero groups:

$$E_1^{1,n-3} = Z, \quad E_1^{1,2n-6} = Z, \quad E_1^{1,2n-5} = Z, \quad E_1^{2,2n-6} = Z+Z+Z, \quad E_1^{3,2n-6} = Z+Z+Z$$

(the difficulty in establishing these relations consists only in computing the groups $E_1^{1,q} = H_{q-n+3}(A_1; Z)$; however, since we are interested only in the homology groups $H_i(A_1; Z)$ in dimensions $\leq 2n - 7$, this difficulty can be overcome by the methods used in the proof of Proposition 1).

Next, knowing the homology groups

$$H_l(Im(S^1, S^n); Z) = H_l(\Omega(S^{n-1}); Z),$$

we obtain information about the groups $E_{\infty}^{p,q}$; namely,

$$E_{\infty}^{p,q} = 0 \quad \text{for } p+q \neq k(n-2),$$

and the groups

$$\sum_{p+q=n-2} E_{\infty}^{p,q} \quad \text{and} \quad \sum_{p+q=2n-4} E_{\infty}^{p,q}$$

are groups associated with the group Z in some filtration. By direct geometric considerations one can compute the differential d_1 on the groups $E_1^{3,2n-6}$, $E_1^{1,n-3}$, and $E_1^{1,2n-5}$; it turns out that

$$d_1 E_1^{3,2n-6} = E_1^{2,2n-6}, \quad d_1 E_1^{1,n-3} = 0, \quad d_1 E_1^{1,2n-5} = 0.$$

From the relation

$$d_1 E_1^{3,2n-6} = E_1^{2,2n-6}$$

it follows, obviously, that

$$d_1 E_1^{2,2n-6} = 0;$$

noting in addition that

$$E_2^{1,2n-6} = E_{\infty}^{1,2n-6} = 0, \quad E_2^{0,2n-6} = E_1^{0,2n-6},$$

we see that the homomorphism

$$d_1 : E_1^{1,2n-6} \rightarrow E_1^{0,2n-6}$$

is an isomorphism. Thus

$$H_{2n-6}(Pl_0) = E_1^{1,2n-6} = Z.$$

From the structure of the differential d_1 described above it is clear that, among the groups $E_2^{p,q}$ with $p > 0$, in the dimensions of interest to us only the groups

$$E_2^{1,n-3} = Z \quad \text{and} \quad E_2^{1,2n-5} = Z$$

are different from zero; on the other hand, evidently,

$$E_1^{0,q} = E_2^{0,q} \quad \text{for } q \neq 2n-6.$$

By dimensional considerations it is clear that

$$E_2^{p,q} = E_{\infty}^{p,q} \quad \text{for } p+q \leq 3n-10.$$

The structure of E_{∞} known to us gives the exact sequence

$$0 \rightarrow E_2^{0,q} \rightarrow H_q(Im) \rightarrow E_2^{1,q} \rightarrow 0,$$

from which it follows that

$$E_2^{0,q} = 0$$

(under the homomorphism Z to Z the kernel is equal to 0). Thus, for

$$q \leq 3n-10, \quad q \neq 2n-6,$$

$$H_q(Pl_0) = E_1^{0,q} = E_2^{0,q} = 0.$$

Proposition 5. *If W^n is a compact simply connected manifold of dimension > 3 , then all homology groups $H_i(Pl_0(S^1, W^n); \mathbb{Z})$ are of finite type.*

This assertion is of some interest, since the group $H_0(Pl_0(S^1, S^3); \mathbb{Z})$ is not of finite type (there exists an infinite set of non-isotopic knots). The proof is based on Proposition 3 (more precisely, on its generalization, in which, by considering spaces of immersions with all possible types of singularities, the dimensional restriction $l \leq 3n - 8$ is removed).

The computation of the homology groups of the space $Pl_0(V^k, S^n)$ can also be carried out by means of a spectral sequence relating the homology groups of $Pl_0(V^k, S^n)$ to the homology groups of $Pl_0(V^k, S^{n+1})$. We shall now describe this spectral sequence in small dimensions. First note that the space S_1^n is homeomorphic to the ball E^n ; therefore there is a natural projection $S_1^{n+1} \rightarrow S_1^n$, inducing a mapping λ of the space $Pl_0(V^k, S^{n+1})$ into the space of mappings $V_1^k \rightarrow S_1^n = E^n$. Using this, we distinguish in the space $Pl_0(V^k, S^{n+1})$ the subspaces $B_0 = \lambda^{-1}(A_0)$, $B_1 = \lambda^{-1}(A_1)$, $B_2 = \lambda^{-1}(R)$, where R is the set of mappings $V_1^k \rightarrow S^n$ having no multiple points and having differential of rank k at all points except one, where the rank of the differential is $k - 1$, and in the direction of degeneration of the first differential the second differential does not degenerate. It is easy to see that B_0 is homotopy equivalent to A_0 , i.e. to $Pl_0(V^k, S^n)$; B_1 is homotopy equivalent to a double covering over A_1 , and B_2 to a double covering over $\lambda(B_2)$.

Considering the spectral sequence associated with the filtration $E_0 = B_0 \subset E_1 = B_0 \cup B_1 \subset E_2 = B_0 \cup B_1 \cup B_2$, we obtain, by arguments entirely analogous to those used in the proof of Proposition 3, the following assertion.

Proposition 6. *If $n \geq 2k + 1$, then there exists a spectral sequence for which*

$$E_1^{0,q} = H_q(Pl_0(V^k, S^n); \mathbb{Z}), \quad E_1^{1,q} = H_{q-n+2k+1}(B_1; \mathbb{Z}),$$

$$E_1^{2,q} = H_{q-n+2k+1}(B_2; \mathbb{Z}), \quad E_1^{p,q} = 0 \quad \text{for } p \geq 3 \text{ and } p < 0,$$

and for $l \leq 2n - 4k - 2$ the group

$$\sum_{p+q=l} E_\infty^{p,q}$$

is a group associated with $H_l(Pl_0(V^k, S^{n+1}); \mathbb{Z})$ for some filtration.

With the help of this proposition a simple proof of Proposition 2 can be given. Namely, in the spectral sequence of Proposition 5 for $V^k = S^k$ the groups $E_1^{p,q} = 0$ for $p > 0$, $q \leq 2n - 4k - 3$, with the exception of the groups

$$E_1^{1,n-2k-1} = E_2^{2,n-2k-1} = E_1^{1,n-k-2} = E_1^{2,n-k-2} = \mathbb{Z}.$$

Geometric arguments show that

$$d_1 E_1^{2,n-2k-1} = E_1^{1,n-2k-1},$$

$$d_1 E_1^{2,n-k-2} = E_1^{1,n-k-2};$$

therefore $E_2^{p,q} = 0$ for all $p > 0$, $q \leq 2n - 4k - 3$. Hence we conclude that

$$H_i(Pl_0(S^k, S^n); \mathbb{Z}) = H_i(Pl_0(S^k, S^{n+1}); \mathbb{Z})$$

for $i \leq 2n - 4k - 3$, i.e.

$$H_i(Pl_0(S^k, S^n); \mathbb{Z}) = H_i(Pl_0(S^k, S^r); \mathbb{Z})$$

for $i \leq 2n - 4k - 3$ and $r > n$. Since, in an obvious way,

$$H_i(Pl_0(S^k, S^r); \mathbb{Z}) = 0$$

for sufficiently large r (see, for example, Proposition 1), Proposition 2 follows.

Moscow Engineering Physics Institute

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REFERENCES

1. A. M. Vinogradov, DAN, **156**, No. 5 (1964).
2. J. Eells, Proc. Nat. Acad. Sci. U.S.A., **45**, 1951 (1959).

Note: Figure translations are in progress. See original paper for figures.

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